UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 9 June, 2011 9:00 am to 12:00 pm

PAPER 69

FLUID DYNAMICS OF ENERGY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

*This is an **OPEN BOOK** examination.*
Candidates may bring handwritten or personally typed lecture notes and handouts only into the examination, no other photocopies of published materials allowed.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Fluid fills a vertical tube of radius r. The fluid is unstably stratified in density and has density gradient $\dfrac{\partial \rho}{\partial z} > 0$, with z pointing upward. Use mixing-length theory to show that the equation governing the evolution of density is

$$\frac{\partial \rho}{\partial t} = \Gamma \frac{\partial}{\partial z}\left[\left(\frac{\partial \rho}{\partial z}\right)^{3/2}\right],$$

where Γ is to be found.

If the tube initially contains fluid of density $\rho + \Delta\rho$ for $0 < z < d$ and fluid of density ρ for $-d < z < 0$, find an analytic solution for the variation of density with time, up to a specific time to be found when the density of the fluid at the top and bottom of the tube begins to change.

[You may assume the flows are of high Reynolds number.]

2

Fluid of density $\rho + \Delta\rho$ migrates along a slope of angle θ through a deep porous medium of porosity ϕ, and permeability K, which is initially filled with fluid of density ρ. The fluid of density $\rho + \Delta\rho$ is supplied from a point source with flux Q. Show that the steady current may be described by an equation of the form

$$\sin\theta \frac{\partial h}{\partial x} = \cos\theta \frac{\partial}{\partial y}\left(h \frac{\partial h}{\partial y}\right),$$

and find a solution for the shape $h(x, y)$, where x is the downslope coordinate and y the cross-slope coordinate.

If fluid drains through a fault in the boundary at a rate proportional to the local depth of fluid of density $\rho + \Delta\rho$, i.e. with flux uh where $u = \dfrac{K\Delta\rho g}{\mu}$, find the maximum change in depth of the current if the fault is of width Δx at a distance x downslope, where $x \gg \Delta x$.

[You may assume there is negligible cross-slope flow as the current migrates over the width of the fault.]

3

Consider the following flow geometry. There is a time-dependent source of actual (specific) buoyancy flux $F_s(t)$ alone at the top (at $z = 0$) of a room of depth H and cross-sectional area A_c which cools (and thus increases the density) of the fluid at $z = 0$. There is an inflow of actual volume flux $Q_T(t) \leqslant 0$ through a top opening with effective area A_T. There is an outflow of actual volume flux $Q_B(t) \geqslant 0$ through a bottom opening at $z = H$. You may assume that density variations are sufficiently small so that the Boussinesq approximation may be used, the pressure is hydrostatic everywhere, and that there is an appropriate reference density ρ_A, the (constant) density of the external ambient.

a) By applying conservation of volume, buoyancy and Bernouilli's equation, show that

$$F_s(t) = g'_R(0,t)Q_B(t) = g'_R(0,t)A_\star \left(\int_0^H g'_R dz \right)^{1/2}, \qquad (*)$$

where $g'_R(z,t)$ is the horizontally-averaged reduced gravity in the room and $A_\star$ is some appropriate area, both of which you should define carefully.

b) Assuming that diffusion is insignificant in the interior of the room, write down a partial differential equation for the evolution of g'_R. You are given that there is initally a stable stratification in the room such that

$$0 < g'_R(0,0) = g_0 < g'_R(H,0) = g_H = g_0(1+\Delta).$$

Show that the reduced gravity in the room may be expressed as a separable function

$$g'_R(z,t) = g_0 e^{\lambda z}\hat{g}(t) = g_0(1+\Delta)^\zeta \hat{g}(t), \qquad (\dagger)$$

where $\zeta = z/H$, and therefore that this model can describe initially linear profiles in density in a particular limit which you should identify.

c) For general profiles defined by ($\dagger$), show that

$$\frac{d}{d\tau}\hat{g} = -\left[\hat{g}^3 \Delta \log(1+\Delta)\right]^{1/2}, \qquad (\ddagger)$$

where time has been scaled with the draining time T_d, so that

$$\tau = \frac{t}{T_d}, \quad T_d = \frac{A_c H}{A_\star (g_0 H)^{1/2}}.$$

Use ($\ddagger$) to calculate the time-dependent form for $g'_R(z,t)$ and $Q_B(t)$, and hence by consideration of ($*$), determine the required time-dependence of the source buoyancy flux $F_s(t)$ for consistency with the separable solution ($\dagger$).

4

a) Write down the standard form of the $k-\epsilon$ turbulence model, quoting the transport equations for the turbulent kinetic energy k and dissipation rate ϵ and the appropriate eddy viscosity hypothesis. Define k, ϵ and P (the term describing the production of turbulent kinetic energy).

b) Consider a turbulent flow which is homogeneous, (i.e. statistically invariant with respect to translations in the reference frame) unsheared, (i.e. there are no externally imposed mean velocity gradients) and unforced. Show that the $k-\epsilon$ equations have decaying solutions

$$k(t) = k_0 \left(\frac{t}{t_0}\right)^{-n}, \qquad \epsilon(t) = \epsilon_0 \left(\frac{t}{t_0}\right)^{-(n+1)}.$$

Express the reference time t_0 in terms of the decay exponent n, ϵ_0 and k_0, and express n in terms of the empirical constants in the $k-\epsilon$ model.

c) Now consider a turbulent shear flow, where the mean shear rate $\mathcal{S}$ has a single nonzero component defined as

$$\mathcal{S} = \frac{\partial}{\partial y}\langle U\rangle > 0,$$

where $\langle . \rangle$ is an appropriate ensemble averaging. It is empirically observed that $\mathcal{S}k/\epsilon$ is constant, where $\mathcal{S}$ is the (imposed) mean shear rate. By considering an equation for the turbulence time scale $\tau = k/\epsilon$, show that P/ϵ is also expected to be constant with time.

d) By considering the eddy viscosity and turbulence production $\mathcal{P}$, show that

$$\frac{|\langle uv\rangle|}{k} = C_\mu \frac{\mathcal{S}k}{\epsilon}, \qquad \frac{\mathcal{P}}{\epsilon} = C_\mu \left(\frac{\mathcal{S}k}{\epsilon}\right)^2, \tag{$*$}$$

where C_μ is the constant relating eddy viscosity to an expression involving k and ϵ. Hence, explain why the empirical observations that $|\langle uv\rangle|/k \simeq 0.3$ and $\mathcal{P} \simeq \epsilon$ suggest that $C_\mu = 0.09$ is a self-consistent choice.

e) Now relax the assumption that $\mathcal{S}k/\epsilon$ is a constant with time, but rather that $\mathcal{S}k/\epsilon$ relaxes to a constant A with the evolution equation

$$\frac{d}{dt}\left(\frac{\epsilon}{k}\right)^2 = -\alpha\frac{\epsilon}{k}\left(\frac{\epsilon^2}{k^2} - \frac{\mathcal{S}^2}{A^2}\right), \tag{$\dagger$}$$

where α is a constant. Using ($*$), derive expressions for α and A in terms of the parameters of the $k-\epsilon$ model so that ($\dagger$) is consistent with the ϵ equation in standard form.

END OF PAPER