UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 9 June, 2011 1:30 pm to 3:30 pm

PAPER 68

FREE BOUNDARY PROBLEMS AND APPLICATIONS

*Attempt question **ONE** and no more than **ONE** other question.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Conisder u_ε to be a solution of the following regularized version of the Obstacle Problem in the unit ball $B_1 := \{x \in \mathbb{R}^n : |x| < 1\}$ in the sense of distributions

$$\begin{cases} \Delta u_\varepsilon = \chi_\varepsilon(u_\varepsilon) & \text{in } B_1 \\ u_\varepsilon = 0 & \text{on } \partial B_1 \end{cases} \tag{1}$$

where $\chi_\varepsilon(s)$ is a smooth approximation of the Heaviside function $\chi_{\{s>0\}}$ such that

$$\chi'_\varepsilon \geqslant 0, \quad \chi_\varepsilon(s) = 0 \text{ for } s \leqslant -\varepsilon, \quad \chi_\varepsilon(s) = 1 \text{ for } s \geqslant \varepsilon.$$

1) Take a direction $e \in \partial B_1$ and prove that for the $\partial_e u$ directional derivative of u_ε the following holds:

$$\int_{B_1} \nabla(\partial_e u_\varepsilon)\nabla\eta dx = -\int_{B_1} \eta\chi'_\varepsilon(u_\varepsilon)\partial_e u_\varepsilon dx \tag{2}$$

for any test function $\eta \in C_0^\infty(B_1)$ (smooth functions with compact support in B_1).

2) Choose $\eta = \partial_e u_\varepsilon \zeta^2$ for some $\zeta \in C_0^\infty(B_1)$ in (2) and show that

$$\int_{B_1} |\nabla(\partial_e u_\varepsilon)|^2\zeta^2 dx \leqslant -\int_{B_1} (\zeta\nabla(\partial_e u_\varepsilon))\cdot(2\partial_e u_\varepsilon\nabla\zeta)\, dx \leqslant \int_{B_1} |\zeta\nabla(\partial_e u_\varepsilon)|2|\partial_e u_\varepsilon\nabla\zeta| dx \tag{3}$$

3) Use Young's Inequality

$$a(2b) \leqslant a^2/2 + 2b^2$$

for the right hand side of (3) to obtain

$$\int_{B_1} |\nabla(\partial_e u_\varepsilon)|^2\zeta^2 dx \leqslant 4\int_{B_1} (\partial_e u_\varepsilon)^2|\nabla\zeta|^2 dx \tag{4}$$

4) Assume the family u_ε is uniformly (in ε) bounded in $W^{1,2}_{\text{loc}}(B_1)$:

$$\|u_\varepsilon\|_{W^{1,2}_{\text{loc}}(B_1)} \leqslant C.$$

What can be concluded from (4): which estimate can this replace in the proof of existence of solutions of Obstacle Problem by regularization?

[*Hint: One can choose the direction e in* (4) *in the basis directions* $e_1, e_2, ..., e_n$ *to obtain* $\partial_e u_\varepsilon = \frac{\partial u_\varepsilon}{\partial x_1}$, $\partial_e u_\varepsilon = \frac{\partial u_\varepsilon}{\partial x_2}$, ... $\partial_e u_\varepsilon = \frac{\partial u_\varepsilon}{\partial x_n}$ *respectively and thus get estimates for the second derivatives.*]

2

Let $u \in L^\infty(D)$ be a nonnegative solution of the classical Obstacle Problem:

$$\Delta u = f(x)\chi_{\{u>0\}} \quad \text{in } D. \tag{1}$$

1) State the regularity of u that is obtained from standard elliptic estimates and the knowledge that $f \in L^\infty(D)$. State the optimal regularity of the solution u of (1).

2) State the Theorem of (at most) Quadratic Growth of u away from a given free boundary point x^0. Write down the corollary to the above theorem, which is used in the proof of optimal regularity of u.

3) Use the comparison principle and Harnack's inequality to prove the Quadratic Growth Theorem.

3

The Problem A was the No-sign Obstacle Problem - the generalized version of the Obstacle Problem where there is no obstacle and the solution is allowed to change its sign.

1) State the formulation of Problem A.

2) State the Non-Degeneracy Lemma for Problem A.

3) Prove the Non-Degeneracy Lemma for Problem A.

END OF PAPER