UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 13 June, 2011 9:00 am to 11:00 am

PAPER 67

ANALYTICAL METHODS FOR BOUNDARY VALUE PROBLEMS AND MEDICAL IMAGING

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $u(x,t)$ solve the following initial-boundary value problem:

$$u_t = u_{xx} + \alpha u_x, \quad 0 < x < \infty, \quad 0 < t < T, \quad \alpha > 0,$$

$$u(x,0) = u_0(x), \quad 0 < x < \infty,$$

$$u_x(0,t) - \beta u(0,t) = g(t), \quad 0 < t < \infty,\ \beta > 0,$$

$$u(x,t) \text{ decays sufficiently fast for all } t \text{ as } x \to \infty,$$

where T is a positive constant, $g(t)$ is a smooth function, $u_0(x)$ decays as $x \to \infty$, and

$$g(0) = \frac{du_0}{dx}(0) - \beta u_0(0).$$

(i) Show that the above PDE can be rewritten in the form

$$\left[e^{-ikx+(k^2-i\alpha k)t}u\right]_t = \left[e^{-ikx+(k^2-i\alpha k)t}(u_x + \alpha u + iku)\right]_x .$$

(ii) Assuming that the solution of the above initial-boundary value problem exists, derive a representation of the solution. [You may use Jordan's lemma without proof.]

(iii) Prove that the solution $u(x,t)$ obtained in (ii) satisfies the boundary condition at $x = 0$.

(iv) Explain why the above problem cannot be solved by a classical x-transform.

2

Assume that there exists a function $q(x,y)$ which satifies the modified Helmholtz equation in the semi-strip $0 < x < \infty$, $0 < y < l$, which decays as $x \to \infty$ for all $0 < y < \infty$, and which is sufficiently smooth all the way to the boundary.

(i) Show that the modified Helmholtz equation

$$\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} - 4\lambda q = 0, \quad \lambda > 0,$$

can be written in the form

$$\left[e^{-\left(ikz + \frac{\lambda}{ik}\bar{z}\right)}(q_z + ikq)\right]_{\bar{z}} + \left[e^{-\left(ikz + \frac{\lambda}{ik}\bar{z}\right)}\left(q_{\bar{z}} + \frac{\lambda}{ik}q\right)\right]_z = 0, \quad k \in \mathbb{C}, \; z = x + iy.$$

(ii) By performing a spectral analysis of the differential form

$$d\left[e^{-\left(ikz + \frac{\lambda}{ik}\bar{z}\right)}\mu(z,\bar{z},k)\right] = e^{-\left(ikz + \frac{\lambda}{ik}\bar{z}\right)}\left[(q_z + ikq)dz - \left(q_{\bar{z}} + \frac{\lambda}{ik}q\right)d\bar{z}\right], \quad k \in \mathbb{C},$$

derive an integral representation for the solution of the modified Helmholtz equation in the interior of the above semi-strip.

3

Let $f(z,\bar{z})$ be a continuously differentiable function for z in the compact domain $D \subset \mathbb{R}^2$, whose smooth boundary is denoted by ∂D.

(i) Derive the Pompeiu formula. [You may use Poincaré's lemma without proof.]

(ii) Let C_ρ denote the disc $|z| \leqslant \rho$ and let m, n be non-negative integers. Use Pompeiu's formula to compute explicitly the integral

$$\frac{1}{2i\pi}\int\int_{C_\rho} \frac{\zeta^m \bar{\zeta}^n}{\zeta - z} d\zeta \wedge d\bar{\zeta}, \quad m > n.$$

END OF PAPER