UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 8 June, 2011 9:00 am to 11:00 am

PAPER 66

REACTION-DIFFUSION EQUATIONS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

On a Hilbert space $\mathcal{H}$ consider the abstract non-linear problem

$$\begin{aligned} \frac{d}{dt}u(t) &= Lu(t) + f(t, u(t)), \qquad t \in (0,T), \quad f : [0,T) \times U \mapsto \mathcal{H}, \qquad (1) \\ u(0) &= u_0, \end{aligned}$$

where $U \subset \mathcal{H}$ is an open subset and $T > 0$.

State assumptions on the operator $L : D(L) \subset \mathcal{H} \mapsto \mathcal{H}$ and on f, which are sufficient to prove existence of a unique, classical, local-in-time (i.e. on a time interval $[0, t_0) \subset [0, T)$ for t_0 small enough) solution for given initial data $u_0 \in \mathcal{H}_\alpha := D((-L)^\alpha)$ for $\alpha \in [0, 1)$. You should give a definition of a classical, local-in-time solution.

Sketch the following central parts of the proof:
With $\mathcal{X} := C\big([0, t_0], \mathcal{H}\big)$ and $\|x\|_{\mathcal{X}} = \max_{0 \leqslant t \leqslant t_0} \|x(t)\|$ for $x \in \mathcal{X}$, define a fixed-point mapping $F : \mathcal{X} \mapsto \mathcal{X}$ and a suitable subset $S \subset \mathcal{X}$, such that (i) F maps S onto S, and (ii) F is a contraction.

2

Consider a dynamical system $\{U_t\}$ in C (a subset of some Banach space) and a stationary point $0 \in C$. Let $V : C \mapsto \mathbb{R}$ be a Lyapunov functional with $V(0) = 0$.

Which properties does V satisfy as a Lyapunov functional?

Show that 0 is stable if $V(u) \geqslant c(\|u\|)$ for $u \in C$, where c is a continuous, strictly monotone increasing function with $c(0) = 0$.

Show moreover that 0 is asymptotically stable in C if additionally $\dot{V}(u) \leqslant -c_1(\|u\|)$, where c_1 has the same properties as c.

3

Consider the equation

$$\partial_t u = \partial_{xx} u + f(u), \quad x \in \mathbb{R}, \; t > 0,$$

where $f : \mathbb{R} \mapsto \mathbb{R}$ is a non-linear function with three zeros on the interval $[0, 1]$, i.e. $f(0) = f(1) = f(x_0) = 0$ with $x_0 \in (0, 1)$. Moreover, assume that $f'(0) < 0$, $f'(x_0) > 0$, $f'(1) < 0$, and that $\int_0^1 f(u)du > 0$.

Construct via phase-plane analysis a travelling wave solution $u(t, x) = w(x - ct)$ with unique wave speed c, which connects $w(-\infty) = 0$ with $w(\infty) = 1$.

[*Hint: draw the phase portrait in the special case* $c = 0$ *and argue the changes for positive waves with speed* $c < 0$.]

4

Consider a travelling wave solution $u = w_c(z)$, where $z = x - ct$ with $c > 0$, $w_c(-\infty) = 1$ and $w_c(\infty) = 0$, of the equation

$$\partial_t u = \partial_{xx} u + f(u), \quad x \in \mathbb{R},\ t > 0,$$

where $f : \mathbb{R} \mapsto \mathbb{R}$ is a non-linear function satisfying $f(0) = 0$, $f'(0) < 0$ and $f(1) = 0$, $f'(1) < 0$.

Consider a small perturbation of the travelling wave, i.e. $u(t, x) = w_c(z) + \epsilon v(t, x)$ and $\epsilon \ll 1$. What sort of stability of travelling waves can be expected? Which linear operator A needs to considered?

Quoting any theorem you rely on, show stability of the essential spectrum with respect to perturbations $v \in L^2$.

What can one say about possible eigenvalues λ satisfying $Av = \lambda v$ with $v \in L^2$? [*Hint: Consider* $\Re(\lambda) \geqslant 0$, *for which* $y(z) = v(z)\, e^{cz/2} \in L^2$ *and use that*

$$\int_{\mathbb{R}} \left(f'(w_c) - \frac{c^2}{4} \right) y^2 dz = - \int_{\mathbb{R}} \left[\left(\frac{y}{y_c} \right)' \right]^2 y_c^2 \, dz + \int_{\mathbb{R}} (y')^2 dz,$$

where $y_c(z) := w_c'(z)\, e^{cz/2}$.]

END OF PAPER