UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 6 June, 2011 9:00 am to 12:00 pm

PAPER 65

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

Attempt no more than ***THREE*** *questions from Section I*
and ***ONE*** *from Section II.*

There are ***SEVEN*** *questions in total.*

The questions in Section II carry twice the weight of those in Section I.
Questions in each Section carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1

Given the ODE system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ where $\mathbf{f} \in \mathrm{C}^\infty$, we can obtain differential equations for higher derivatives, $\mathbf{y}^{(k)} = \mathbf{f}_k(\mathbf{y})$, $k = 0, 1, \ldots$, by repeated differentiation. Suppose that the ODE is solved by the one-step, multiderivative method

$$\sum_{k=0}^{n} q_k h^k \mathbf{f}_k(\mathbf{y}_{n+1}) = \sum_{k=0}^{m} p_k h^k \mathbf{f}_k(\mathbf{y}_n),$$

where $m, n \geqslant 0$ are integers and the q_k and p_k are given coefficients, with $q_0 = 1$. Let

$$r(z) = \frac{\sum_{k=0}^{m} p_k z^k}{\sum_{k=0}^{n} q_k z^k}.$$

1. Prove that the method is of order p if and only if $r(z) = \mathrm{e}^z + O(z^{p+1})$, $z \to 0$.
2. Determine the conditions on the function r that ensure A-stability of the underlying method.
3. Determine whether the method of order $m + n$ is A-stable for (a) $m = 0$, $n = 2$; and (b) $m = 0$, $n = 3$.

2

Consider the two-step method

$$\mathbf{y}_{n+2} - (1 + a)\mathbf{y}_{n+1} + a\mathbf{y}_n = \frac{h}{2}[(1 + a)\mathbf{f}(\mathbf{y}_{n+2}) + (1 - 3a)\mathbf{f}(\mathbf{y}_{n+1})],$$

and $a \in \mathbb{R}$ is a parameter, for the solution of the ODE $\mathbf{y}' = \mathbf{f}(\mathbf{y})$.

1. Determine the order of the method.
2. For which values of a is the method convergent?
3. For which values of a is the method A-stable?

3

Let a be a smoothly differentiable function such that $a(x,y) > 0$, $x, y \in [0,1]$ and set

$$\mathcal{L}u = -\nabla^\top(a\nabla u), \qquad x, y \in [0,1].$$

1. Prove that $\mathcal{L}$ is a positive definite operator on a Hilbert space $\mathcal{H}$ (which you should identify) and, quoting all relevant results, state the variational problem corresponding to the differential equation

$$\mathcal{L}u = f, \qquad x, y \in [0,1],$$

given with zero Dirichlet boundary conditions.

2. Let $\mathcal{H}_M$ be an M-dimensional subspace of $\mathcal{H}$ spanned by the basis $\{\varphi_1, \varphi_2, \ldots, \varphi_M\}$. Derive the linear equations that need be solved once the differential equation is discretized with the Ritz method in this basis.

3. Propose an appropriate choice of a finite element basis $\{\varphi_1, \varphi_2, \ldots, \varphi_M\}$.

4

We are solving the advection equation $u_t = u_x$ with the two-step method

$$u_m^{n+1} = \alpha_0 u_m^n + \alpha_1 u_{m+1}^n + \beta_0 u_m^{n-1}.$$

1. Find coefficients $\alpha_0, \alpha_1, \beta_0$ so that the method is of order 2.

2. The equation is given for $x \in [-1,1]$ with periodic boundary conditions. Prove that, with the coefficients that you have derived in Part 1, the method cannot be stable for any $\mu \in (0,1)$, where μ is the Courant number.

5

Consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2\alpha \frac{\partial u}{\partial x}, \qquad t \geqslant 0, \quad x \in [-1, 1],$$

where $u = u(x,t)$ and $\alpha \in \mathbb{R}$, given with an initial condition for $t = 0$ and zero Dirichlet boundary conditions at $x = 0, 1$.

1. Prove that the equation is well posed in the standard $L_2[0,1]$ norm.

2. The equation is semi-discretized by the central difference approximation

$$u_m' = \frac{u_{m-1} - 2u_m + u_{m+1}}{(\Delta x)^2} + \alpha \frac{u_{m+1} - u_{m-1}}{\Delta x}, \qquad m = 1, 2, \ldots, M,$$

   where $\Delta x = 1/(M+1)$. Carefully justifying all steps, prove that the semi-discretization converges to the exact solution of the differential equation in every compact time interval.

SECTION II

6

Write an essay on collocation methods for ordinary differential equations.

7

Write an essay on stability analysis of numerical methods for partial differential equations of evolution using eigenvalue analysis.

END OF PAPER