UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 9 June, 2011 9:00 am to 11:00 am

PAPER 63

DYNAMICS OF ASTROPHYSICAL DISCS

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Accretion on to a black hole can be modelled within Newtonian dynamics by using the modified gravitational potential

$$\Phi = -\frac{GM}{R - R_{\mathrm{s}}} \,,$$

where $R = (r^2 + z^2)^{1/2}$ is the spherical radial coordinate in terms of cylindrical polar coordinates (r, ϕ, z), and R_{s} is the radius of the event horizon of the black hole.

By constructing an effective potential governing the r and z components of the equation of motion, or otherwise, determine the orbital frequency $\Omega(r)$, the epicyclic frequency $\kappa(r)$ and the vertical frequency $\Omega_z(r)$ of circular orbits of radius $r > 3R_{\mathrm{s}}$ in the plane $z = 0$, in this potential. Explain the significance of the result $\Omega_z = \Omega$. Show that circular orbits are unstable for $1 < r/R_{\mathrm{s}} < 3$. Describe briefly the transition that occurs in an accretion disc at $r = 3R_{\mathrm{s}}$.

(b) The surface density of a Keplerian accretion disc is governed by the diffusion equation

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2} \frac{\partial}{\partial r}(r^{1/2} \bar{\nu} \Sigma) \right] .$$

Assuming that the mean effective kinematic viscosity $\bar{\nu}$ is a non-zero constant, find all solutions of this equation of the form

$$\Sigma \propto r^a t^b \exp\left(-\frac{cr^2}{\bar{\nu} t} \right) ,$$

valid for $t > 0$, where a, b and c (with $c > 0$) are constants to be determined. Do these solutions conserve the total mass or the total angular momentum of the disc? Discuss briefly the interpretation of these solutions.

2

The Navier–Stokes equations for a homogeneous incompressible fluid in a rotating frame of reference are

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\boldsymbol{\nabla}\Phi - \frac{1}{\rho}\boldsymbol{\nabla} p + \nu \nabla^2 \boldsymbol{u}\,,$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0\,.$$

The velocity field in the local approximation may be written as

$$\boldsymbol{u} = -Sx\,\boldsymbol{e}_y + \boldsymbol{v}\,,$$

where S is a constant and $\boldsymbol{v}$ is the velocity perturbation (which need not be small).

(i) For a two-dimensional flow in the (x, y) plane, independent of z, show that $\boldsymbol{v}$ can be derived from a streamfunction ψ satisfying

$$\frac{\partial \zeta}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \zeta = \nu \nabla^2 \zeta\,, \tag{1}$$

where $\zeta = -\nabla^2 \psi$ is the vorticity perturbation.

(ii) Show that solutions of equation (1) exist in the form of shearing waves,

$$\psi = \mathrm{Re}\left\{\tilde{\psi}(t) \exp[\mathrm{i}\boldsymbol{k}(t) \cdot \boldsymbol{x}]\right\}\,,$$

and find the equations governing the evolution of the amplitude $\tilde{\psi}$ and the wavevector $\boldsymbol{k}$. Determine the dependence of the kinetic energy of non-axisymmetric disturbances on time, and describe this qualitatively in the cases $\nu = 0$ and $\nu > 0$. Describe briefly how this behaviour compares with that of three-dimensional axisymmetric disturbances in non-rotating and (Rayleigh-stable) rotating shear flows.

3

This question is about a mechanical analogue of the magnetorotational instability.

In the local approximation, the dynamics of two particles of mass m connected by a spring of spring constant $k = \beta m$ is described by the equations

$$\begin{aligned} \ddot{x}_1 - 2\Omega\dot{y}_1 - 2\Omega S x_1 &= \beta(x_2 - x_1)\,, \\ \ddot{y}_1 + 2\Omega\dot{x}_1 &= \beta(y_2 - y_1)\,, \\ \ddot{z}_1 + \Omega_z^2 z_1 &= \beta(z_2 - z_1)\,, \end{aligned}$$

together with similar equations in which the suffixes 1 and 2 are interchanged.

Give a physical interpretation of the equations, explaining the meaning of the symbols Ω, S and Ω_z.

Assume that the quantities β, Ω, S, $\kappa^2 = 2\Omega(2\Omega - S)$ and Ω_z^2 are positive. Show that relative motions of the two particles in the (x, y) plane proportional to $\exp(\lambda t)$ are possible, where

$$\lambda^4 + (\kappa^2 + 4\beta)\lambda^2 + 4\beta(\beta - \Omega S) = 0\,.$$

Determine the range of β for which instability occurs. For fixed Ω and S, find the maximum growth rate of the instability and the value of β for which this occurs. Write down the explicit form of $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$ for this optimal solution.

Discuss the relation of this problem to the magnetorotational instability in astrophysical discs. In the magnetohydrodynamic case, what quantity would correspond to β in the above analysis? How is the optimization of the growth rate with respect to β achieved in the magnetohydrodynamic case?

END OF PAPER