UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 9 June, 2011 1:30 pm to 3:30 pm

PAPER 62

BINARY STARS

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A red giant of mass M_1 has a main-sequence companion of mass M_2 in a circular orbit. The red giant is losing mass in a fast, isotropic, spherically symmetric wind at a rate $\dot{M}$. Show that, if the intrinsic spin angular momentum of the stars can be neglected, the orbital angular momentum $J_{\rm orb}$ obeys

$$\frac{\dot{J}_{\rm orb}}{J_{\rm orb}} = \frac{M_2 \dot{M}}{M_1 M},$$

where $M = M_1 + M_2$. Thence, assuming tides are strong enough to keep the orbit circular, show that the orbital period P and separation a change according to

$$P \propto M^{-2}, \quad a \propto M^{-1}.$$

The radius R_1 of the giant responds to changes in mass according to

$$R_1 \propto f(L) M_1^{-n}, \quad 0 < n < 1,$$

where L is its luminosity that is determined by conditions in its core. The radius $R_{\rm L}$ of its Roche lobe is approximated by

$$\frac{R_{\rm L}}{a} = 0.426 \left(\frac{M_1}{M}\right)^{\frac{1}{3}}.$$

Now suppose that the giant is filling its Roche lobe and that mass is lost in the wind on a timescale much shorter than the giant's nuclear evolution timescale. Show, by differentiating $\log(R_1/R_{\rm L})$ or otherwise, that mass transfer is driven by the wind if

$$q = \frac{M_1}{M_2} < \frac{1+3n}{3(1-n)}. \qquad (*)$$

What happens otherwise?

Show further that, when $(*)$ is satisfied and $6q < 5 - 3n$, the rate of conservative mass transfer to the main-sequence star

$$\dot{M}_2 = -\frac{1 + 3n - 3(1-n)q}{(1+q)(5 - 3n - 6q)} \dot{M}.$$

What is the physical consequence if $6q > 5 - 3n$?

2

A binary system of total mass M consists of two stars of masses M_1 and M_2 in an elliptical orbit and sufficiently separated that they behave as point mass objects. Show that the energy of the orbit

$$E = \mu\left(\frac{1}{2}v^2 - \frac{GM}{r}\right),$$

where $\mu = M_1M_2/M$ is the reduced mass, r and v are the instantaneous separation and relative speed of the stars and G is Newton's constant. Show further that

$$E = -\frac{GM\mu}{2a} = \text{const},$$

where $l = a(1-e^2)$ is the semi-major axis of the orbit parametrised by

$$r = \frac{l}{1 + e\cos\theta},$$

with $0 \leqslant \theta < 2\pi$.

In a system with a circular orbit the star 1 of mass M_1 undergoes a supernova explosion to leave a neutron star of mass M_1' in a time much shorter than the orbital period. The mass lost to the gravitational binding energy of the neutron star may be neglected. The matter is ejected in an asymmetric manner such that the neutron star experiences a kick of speed $u = \alpha v$ at an angle θ to the orbital motion of star 1. Show that the new total mass M' and semi-major axis a' are related by

$$\frac{M'}{a'} = \frac{2M'}{a} - (1 + 2\alpha\cos\theta + \alpha^2)\frac{M}{a}$$

and thence that the system becomes unbound if

$$M' < \frac{1}{2}(1+\alpha)^2 M$$

but could remain bound if

$$M' > \frac{1}{2}(1-\alpha)^2 M.$$

If the system becomes unbound show that the stars eventually recede from one another at a speed

$$V = \left\{(1 + 2\alpha\cos\theta + \alpha^2) - \frac{2M'}{M}\right\}^{\frac{1}{2}} v.$$

Suppose the kick is at an angle ϕ, where $-\pi/2 < \phi < \pi/2$, to the plane of the original orbit. By considering the change in specific angular momentum $\boldsymbol{h}$ show that the i inclination of the new orbit to the original is given by

$$\cos i = \frac{1 + \alpha\cos\theta}{\left[\alpha^2\sin^2\phi + (1+\alpha\cos\theta)^2\right]^{1/2}}.$$

In some models of supernovae any kick is always along the spin axis of the progenitor star. Assuming that the spin of the progenitor was aligned with the original orbit what does this mean for ϕ, θ and for the inclination of the new orbit? If a system is to remain bound in this case show that

$$i < \cos^{-1}\sqrt{\frac{M}{2M'}}.$$

3

A galaxy is 10 Gyr old and has been forming stars at a constant rate since its birth. Show that the fraction of stars younger than t is

$$Y(t) = 0.1\left(\frac{t}{\mathrm{Gyr}}\right).$$

Stars in the galaxy form with masses M chosen from an initial mass function

$$n(M)\,dM = \begin{cases} k\left(\frac{M_\odot}{M}\right)^3 dM, & M > 0.2\,M_\odot, \\ 0, & M < 0.2\,M_\odot, \end{cases}$$

where $n(M)\,dM$ is the number of stars with masses between M and $M + dM$ and k is a constant. Show that the fraction X of stars with mass greater than M is

$$X(M) = \frac{0.04}{(M/M_\odot)^2}, \quad M > 0.2\,M_\odot.$$

A star of mass M spends $8\,\mathrm{Gyr}/(M/M_\odot)^2$ on the main sequence and then a further $2\,\mathrm{Gyr}/(M/M_\odot)^2$ as a red giant before becoming a white dwarf.

On a sketch of the (M, t) plane indicate the area that contains stars which are currently red giants. Show that this maps on to a triangle in the (X, Y) plane and hence show that 0.5% of stars are currently giants and that 2% are currently white dwarfs.

All stars in the galaxy actually form in binary systems with the masses of the two components chosen independently from the above distribution. By considering the subdivisions of a cube or otherwise show that

(i) just under 1% of systems contain a red giant,

(ii) just over 3% of systems containing a red giant also contain another evolved star, either another red giant or a white dwarf and

(iii) for every binary containing two red giants there are eight containing both a white dwarf and a red giant and sixteen containing two white dwarfs.

A species has evolved on a planet in the galaxy to the point where they have built telescopes capable of observing such systems in a local neighbourhood that is small compared with any galactic structure. What effects would they need to take into account when comparing the above fractions with their observations?

Briefly describe how binary interaction could affect the numbers?

[The volume of a pyramid is one third of the area of its base times its height.]

END OF PAPER