UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 10 June, 2011 9:00 am to 12:00 pm

PAPER 60

GALAXIES

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

This is an ***OPEN BOOK*** *examination.*
Candidates may bring handwritten or personally typed lecture notes and handouts only into the examination, no other photocopies of published materials allowed.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Physical Constants

Newtonian gravitational constant	G	$= 6.673 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$
proton rest mass	m_p	$= 1.673 \times 10^{-27}\,\mathrm{kg}$ $= 938\,\mathrm{MeV}$
parsec	1 pc	$= 3.086 \times 10^{16}\,\mathrm{m}$
solar mass	$1\,\mathrm{M}_\odot$	$= 1.989 \times 10^{30}\,\mathrm{kg}$
solar radius	$1\,\mathrm{R}_\odot$	$= 6.960 \times 10^{8}\,\mathrm{m}$
solar luminosity	$1\,\mathrm{L}_\odot$	$= 3.90 \times 10^{26}\,\mathrm{W}$

1

(i) M33 (the Triangulum galaxy), which has a dynamical mass (including baryons and dark matter) of about 5×10^{10} $M_\odot$, is observed to be interacting with Andromeda (M31) through the detection of stellar tidal tails about M33, extending into a stellar bridge to M31. M33 can thus be approximated as a satellite in orbit about M31 at a distance of 180 kpc. Assuming that Andromeda's dark-matter halo has an NFW (Navarro, Frenk, White) density profile and a roughly flat rotation curve (v_c=250 km/s) extending out to M33, estimate how much time it will take for M33 to spiral into the Andromeda (derive appropriate values in the expression for dynamical friction).

Figure 1 shows the most likely orbit that can produce tidal features similar to the M33 observations, which has a pericentric radius $<$ 50 kpc and lies in a plane nearly parallel to the line of sight. Compare your estimate above with this recent model of the interaction between M33 and M31. Describe why the two timescales may be different.

(ii) By equating the cooling timescale to the free-fall timescale, show that the maximum mass of a protogalactic nebula is given by

$$M = 25/32 \frac{\Lambda^2}{G^3 \mu^4 m_H^4 R}.$$

Estimate the maximum mass of a protogalactic nebula that can undergo a free-fall collapse if $R = 60$ kpc. Assume that $\Lambda \sim 10^{-37} W m^3$, with the mean molecular mass, $\mu = \frac{1}{X_H + X_{He}}$.

(iii) M87 is a giant elliptical galaxy near the centre of the Virgo cluster of galaxies, $\sim$17 Mpc distant. It is estimated that its mass is approximately 3×10^{13} $M_\odot$ within a radius of 300 kpc. How long would it take for a star near the outer edge of the galaxy to orbit the centre once?

Compare your answer to the approximate age of our Galaxy.

(iv) Based on the result from part (iii) above, and assuming that M87 has been capturing smaller satellite galaxies up until the present time, would you expect the outer portions of M87 to be in virial equilibrium? Why or why not?

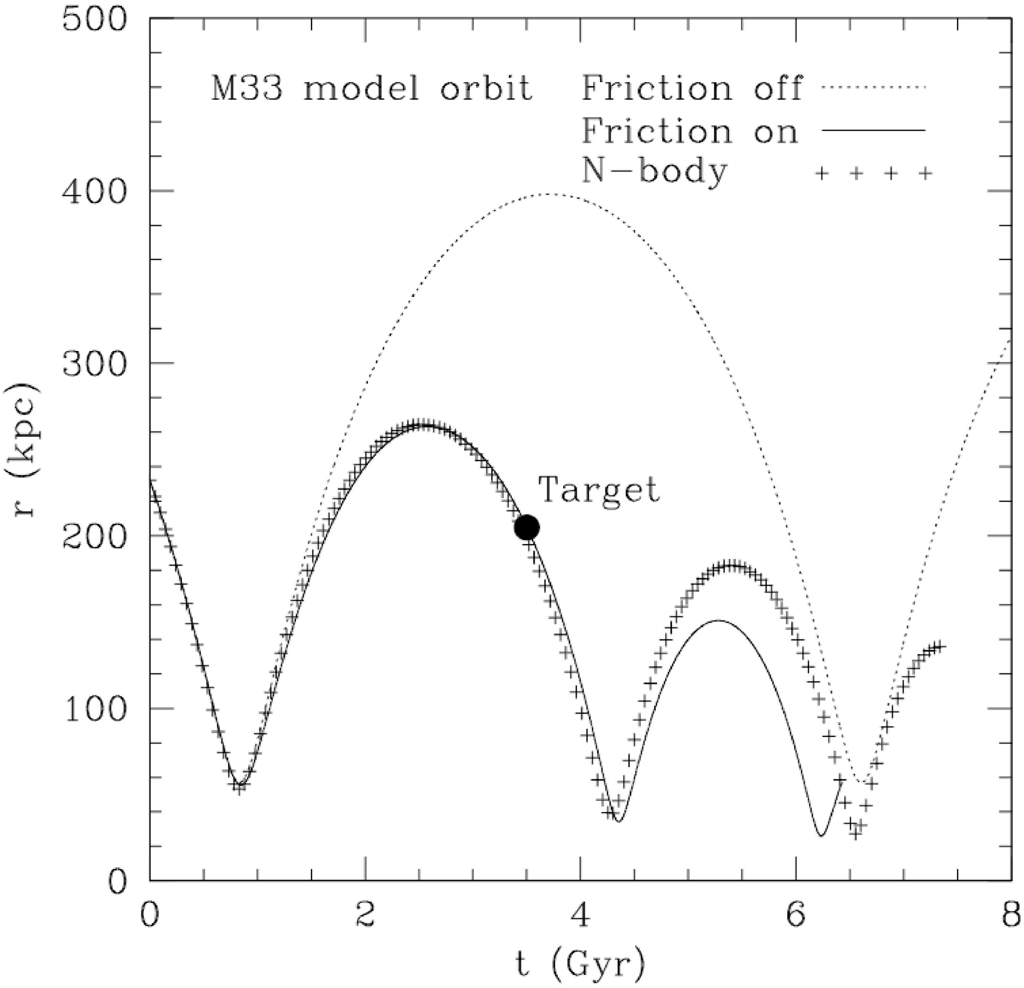

Figure 1: model for orbit of M33 in the potential of M31 (from Dubinski et al. 2011)

2

When a small object (a satellite galaxy or a star) approaches a much more massive object (a host galaxy or a supermassive black hole), the smaller object can be tidally disrupted. The distance of closest approach before being tidally disrupted is the Roche limit. If the small object is a star and the large object is a supermassive black hole at the centre of a large galaxy, the Roche limit is given by

$$r_R = 2.4 \left(\frac{\rho_{\rm BH}}{\rho_*} \right)^{1/3} R_S,$$

where $R_S = 2GM_{\rm BH}/c^2$ is the Schwarzschild radius, $\rho_{\rm BH}$ is the density of the black hole, and ρ_* is the average density of the star.

(i) Derive the Roche limit given above by considering the following argument. Consider a star of mass M_* and radius R_* to be passing a massive object (mass M, radius R) at a distance r (see Figure 2). The star will be tidally disrupted if the difference in the gravitational pull exerted by the massive object on the two halves of the star is greater than the internal gravitational force holding the star together.

(ii) Setting the average density of the supermasssive black hole equal to its mass divided by the volume contained within the Schwarzschild radius, derive an expression for the mass of a black hole that would have $r_R = R_S$.

(iii) If the Sun were to fall into a supermassive black hole, what maximum mass could the black hole have if the Sun would be tidally disrupted before crossing the event horizon? Compare your answer to the mass estimates of typical supermassive black holes in galactic nuclei (i.e., Milky Way: 4×10^6 $\rm M_\odot$; M31: $\sim 10^8$ $\rm M_\odot$; distant active galaxies: $10^6 - 10^{10}$ $\rm M_\odot$)

(iv) If the supermassive black hole exceeded the mass found in part (iii), what would be the implications in terms of liberating the gravitational potential energy of the infalling star? Could infalling stars effectively power active galaxies (assuming an typical active nucleus will have a larger luminosity than the star light from the host galaxy)?

[The density of the sun is given as $\rho_\odot = \frac{3}{4\pi} M_\odot R_\odot^3 \sim 1400$ kg/m^3.]

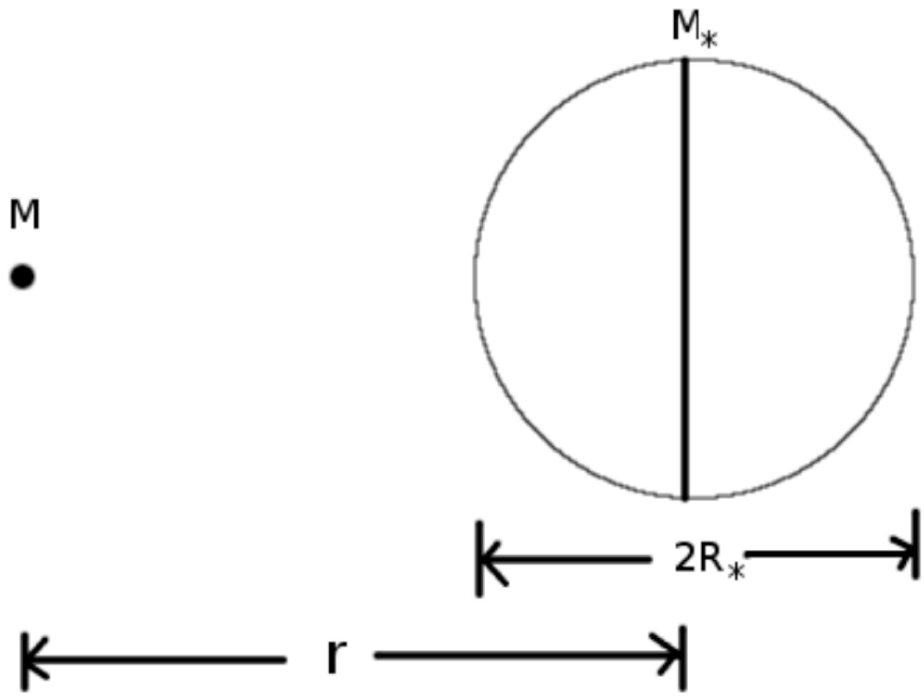

Figure 2: depiction of the small and large mass objects for part (i).

3

The threshold for collapse in the spherical collapse model is a function of cosmological parameters and redshift, and has a value $\delta_c \sim 1.69$. In order to explore when galaxy formation can proceed efficiently and where feedback can be effective in expelling baryons, we will calculate the properties of gas cooling in halos at $z \sim 11$ (small halos) and $z \sim 3$ (large halos, or normal galaxies).

(i) At $z = 11$, 2σ fluctuations which have just collapsed and virialized have scale sizes approximately, $R = 0.1$ Mpc, whereas at $z = 3$, 2σ fluctuations which have just collapsed and virialized have approximately, $R = 2.7$ Mpc. At both epochs, calculate their masses, virial radii, circular velocities, and virial temperatures.

For the mass, you may assume $M = \frac{4\pi}{3}\rho^0_{crit}R^3\Omega^0_m \sim 1.5 \times 10^{11}$ $\mathrm{M}_\odot$ R^3, where R is in units of Mpc. The virial radius, r_{vir} can be derived by considering that $M = \frac{4\pi}{3}\rho_{crit}(a)r^3_{vir}\Delta_{vir}$, where $\Delta_{vir} = \rho_{vir}/\bar{\rho}$ is the virial over-density factor derived in class. At high$-z$, $\rho_{crit} \sim \rho^0_m/a^3 = \Omega^0_m\rho^0_{crit}(1+z)^3$.

Circular velocity you can relate to the potential of the virialized halo, and the viral temperature is related to the kinetic energy of primordial baryons with mass μm_p, with $\mu = 0.6$ the mean molecular weight for primordial gas, and the proton mass $m_p = 1.67 \times 10^{-27}$kg (the Boltzmann constant is 1.38×10^{-23} m^2 kg s^{-2} K^{-1}.

(ii) Using the cooling curve shown in Figure 3 (hydrogen and helium curves only for primordial gas), estimate the ratio of cooling time, $t_{cool} = E(dE/dt)^{-1}$, to collapse time for these objects (assuming that the initial temperature is the virial temperature).

$$E = \frac{5}{2}kT\frac{\rho}{\mu m_p}$$

is the total energy per volume of ideal gas at constant pressure. Why do we assume pressure is constant?

With Λ, the cooling rate for a given element, in erg cm^3 s^{-1},

$$\frac{dE}{dt} = n_e n_p \Lambda_H + n_e n_{He}\Lambda_{He} = \left(\frac{\rho}{m_p}\right)^2\left(\frac{1}{2}X_p + \frac{1}{2}\right)\left[X_p\Lambda_H + \frac{1}{4}(1 - X_p)\Lambda_{He}\right]$$

What is an appropriate hydrogen mass fraction, X_p?

Note that only the baryons participate in cooling, so the density ρ in the equation for t_{cool} can be approximated as the primordial baryon fraction,

$$\frac{\rho_b}{m_p} = \frac{M \times (\Omega_b/\Omega_m)}{(4\pi/3)r^3_{vir}m_p}.$$

At each epoch, do you expect collapse and fragmentation or not?

(iii) With a standard initial mass function, every 100 solar masses of gas turned into stars creates one supernova, with kinetic energy $\sim 10^{51}$ erg. What fraction, f, of the gas in these halos at $z = 11$ and $z = 3$ needs to be converted to stars to produce enough kinetic energy to eject all the rest of the baryons?

What can you conclude about the efficiency of star formation, and galaxy formation in general, at the two epochs?

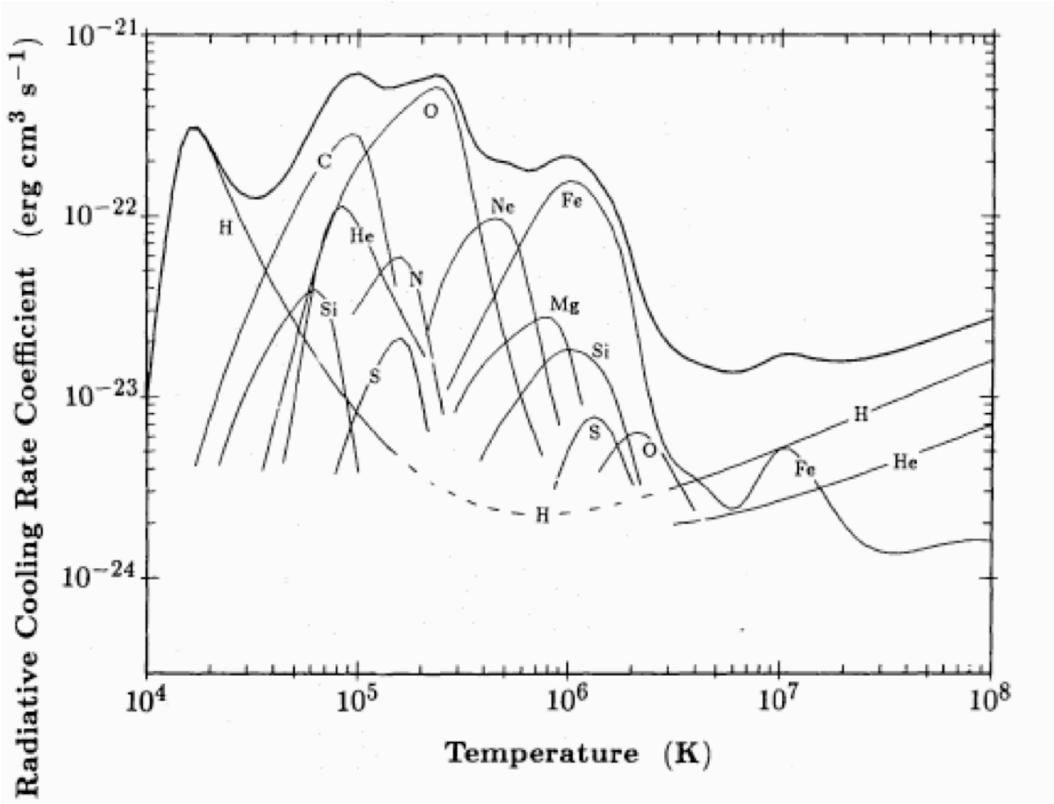

Figure 3: Cooling curves for different elements, for Question 3 Part (ii).

[hint: If a fraction f of the baryons are in stars, the total binding energy of the free baryons is $M_b(1-f)(v_c^2)/2$, which can be compared to the total energy released by supernovae.]

4

As in class notes, the equation for the growth of density perturbations $\delta = \delta\rho/\rho$ in an expanding universe is given as

$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} = \frac{c_s^2}{a^2}\nabla^2\delta + 4\pi G\rho\delta \tag{1}$$

(note that the time derivatives are at fixed comoving positions, and the Laplacian is with respect to comoving coordinates; ρ is the unperturbed density).

Consider a flat universe filled with two components: (1) uniform radiation, and (2) nonuniform cold dark matter ($c_s = 0$) which does not interact with the radiation except via the Hubble expansion:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{4\pi}{3}G\rho_{eq}\left[\left(\frac{a_{eq}}{a}\right)^3 + \left(\frac{a_{eq}}{a}\right)^4\right],$$

where eq denotes matter-radiation equality.

(i) Let $y = \rho_m/\rho_r = a/a_{eq}$, and eliminate the time derivatives in favour of y derivatives in equation (1).

(ii) Recall that $\ddot{a}/a = -4\pi/3G(\rho + 3p/c^2)$, and express this in terms of y and $\dot{a}/a$.

(iii) Since only the cold dark matter is participating in the perturbations, only the matter density enters in the $4\pi G\rho$ on the right-hand side of Eq. (1). Show that therefore $4\pi G\rho_m = (3/2)(\dot{a}/a)^2 y/(1+y)$.

(iv) Combine parts (i-iii) to arrive at the evolution equation for the dark matter perturbation:

$$y(1+y)\frac{\partial^2\delta}{\partial y^2} + (1 + \frac{3}{2}y)\frac{\partial\delta}{\partial y} = \frac{3}{2}\delta \tag{2}$$

(v) Show that one solution of Eq. (2) is $\delta_1(y) = 1 + (3/2)y$.

(vi) As in class notes, the general solution is of the form

$\delta(y) = c_1\delta_1(y) + c_2\delta_2(y)$. A second solution is given as

$$\delta_2(y) = 3\sqrt{1+y} + (1 + \frac{3}{2}y)\ln\left(\frac{\sqrt{1+y}-1}{\sqrt{1+y}+1}\right)$$

Show that during the radiation-dominated epoch ($a \ll a_{eq}$), δ_1 is frozen, while δ_2 grows logarithmically, and during the matter-dominated epoch ($a \gg a_{eq}$), δ_1 grows linearly with y (and hence linearly with redshift).

5

Write an essay on the semi-analytic approach to modeling galaxy formation. Your essay should address the following questions:

What is the basic framework for semi-analytic models as opposed to hydrodynamical simulations? What are the starting points, and what approximations are typically adopted in setting up the initial conditions of the simulation?

What are the key physical processes which are approximated by empirical and semi-analytic recipes?

How is star formation typically treated, and why can star formation not be represented from first principles?

What are the main successes of the model?

Where does the model run into difficulty at present, and are there possible solutions?

What will it take to completely solve galaxy formation and evolution?

END OF PAPER