UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 6 June, 2011 1:30 am to 4:30 pm

PAPER 6

SEMIGROUPS OF OPERATORS

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Suppose that $(T_t)_{t>0}$ is a contraction semigroup acting on a Banach space E, with infinitesimal generator Z. Define the *Laplace transform* L_λ, for $\lambda = \mu + i\nu$, where $\mu > 0$. Show that L_λ is a bounded linear operator on E which commutes with T_t and find an upper bound for its norm.

Show that L_λ is injective, and determine its image $L_\lambda(E)$. Explain the sense in which it commutes with Z.

Show that Z is a closed linear operator, and that $D(Z)$ is dense in E.

Show that Z determines the semigroup $(T_t)_{t>0}$.

2

Suppose that A is a closed linear operator on a Hilbert space H with dense domain $D(A)$. Define the *adjoint* of A.What does it mean to say that A is *symmetric*? What does it mean to say that A is *self-adjoint*?

Suppose that A is symmetric. Show that A is self-adjoint if and only if its spectrum $\sigma(A)$ is contained in the real line.

Suppose that A is self-adjoint. Show that A is positive semi-definite if and only if $\sigma(A) \subseteq [0, \infty)$.

[You may assume that if T is a bounded self-adjoint operator then $\inf\{\langle T(x), x\rangle : \|x\| = 1\}$ is an approximate eigenvalue for T.]

Suppose that $(T_t)_{t\geqslant 0}$ is a contraction semigroup of self-adjoint operators. Show that each T_t is positive definite.

3

Suppose that $(P_t)_{t \geqslant 0}$ is a reversible Feller semigroup with invariant probability distribution μ, infinitesimal generator L, transition probability measures $p_t(x, dy)$ and standard algebra A. Suppose that $f, g \in A$. Define the *squared gradient operator* $\Gamma(f, g)$ and the *joint energy* $\mathcal{E}_\mu(f, g)$.

Suppose that $f \geqslant 0$. Define the *entropy* $\mathrm{Ent}_\mu(f)$. Let $(f \vee \epsilon)(x) = \max(f(x), \epsilon)$ for $0 < \epsilon < 1$. Show that $\mathrm{Ent}_\mu(f \vee \epsilon) \to \mathrm{Ent}_\mu(f)$ as $\epsilon \to 0$.

What does it mean for μ to satisfy a *logarithmic Sobolev inequality* with constant c_{LS}?

Find an expression for $\Gamma(f, g)$ in terms of the transition probability measures. Hence or otherwise show that if $f \geqslant 0$ then $2\mathcal{E}_\mu(f, f) \leqslant \mathcal{E}_\mu(f^2, \log f)$.

Suppose that μ satisfies a logarithmic Sobolev inequality with constant c_{LS}, that $f \in A$ and that $f \geqslant 0$. Show that

$$\mathrm{Ent}_\mu(P_t(f)) \leqslant e^{-4t/c_{LS}} \mathrm{Ent}_\mu(f).$$

4

Define the Bernoulli random variables and the Walsh functions on the hypercube D_2^d, and establish their properties relating to the group structure of D_2^d. By considering properties of a certain linear operator on $L^2(D_2^d)$, show that if $a_1, \ldots, a_d$ are vectors in a normed space E and $\epsilon_1, \ldots, \epsilon_d$ are Bernoulli random variables then

$$\left\| \sum_{i=1}^{d} \epsilon_i a_i \right\|_2 \leqslant \sqrt{2} \left\| \sum_{i=1}^{d} \epsilon_i a_i \right\|_1 .$$

5

Let γ denote standard Gaussian measure on $\mathbf{R}$. Define the creation operator a^+ and the annihilation operator a^-, and show that a^- is the adjoint of a^+. (You may assume that a^+ and a^- are closed, with dense domains.)

Explain how the creation operator is used to define the Hermite polynomials $(h_n)_{n=0}^\infty$, and show that they form an orthogonal sequence in $L^2(\gamma)$. How does the annihilation operator act on h_n?

Define the number operator N, and show that it is positive semi-definite. Why is it called the number operator? Explain how it is used to define the Ornstein-Uhlenbeck semigroup. (You may assume that the polynomial functions are dense in $L^2(\gamma)$.)

Find an expression for the energy $\mathcal{E}_\gamma(f)$ for a differentiable function f with bounded derivative.

The measure γ satisfies a logarithmic Sobolev inequality with constant 2. Use this to show that if f is a continuously differentiable function with $\|f'\|_\infty \leqslant 1$ and if $\int_{-\infty}^{\infty} f(x)\,dx = 0$ then

$$\gamma(\{x \in \mathbf{R} : f(x) > r\}) \leqslant e^{-r^2/2} \quad \text{for } r > 0.$$

END OF PAPER