UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 6 June, 2011 1:30 pm to 4:30 pm

PAPER 59

ASTROPHYSICAL FLUID DYNAMICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

You are reminded of the equations of ideal magnetohydrodynamics in the form

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u}$$

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p = -\gamma p \nabla \cdot \mathbf{u}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\rho \nabla \Phi - \nabla p + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})$$

$$\nabla^2 \Phi = 4\pi G \rho$$

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Starting from the equations of ideal magnetohydrodynamics, show that the equation of motion of a self-gravitating gas with no external gravitational sources moving under a magnetic field $\mathbf{B}$ and its pressure, p may be written in the form

$$\rho \frac{Du_i}{Dt} = \frac{\partial T_{ij}}{\partial x_j},$$

where the summation convention has been used and

$$T_{ij} = D_{ij} + M_{ij}$$

for $i, j = 1, 2, 3$ are the components of a symmetric stress tensor with

$$M_{ij} = \frac{1}{\mu_0}\left(B_i B_j - \frac{|\mathbf{B}|^2}{2}\delta_{ij}\right), \text{ and } D_{ij} = -p\delta_{ij} - \frac{1}{4\pi G}\left(g_i g_j - \frac{|\mathbf{g}|^2}{2}\delta_{ij}\right).$$

The components of $\mathbf{g}$ are $g_i = -\partial\Phi/\partial x_i$, for $i = 1, 2, 3$.

Prove the tensor virial theorem in the form

$$\frac{1}{2}\frac{d^2 I_{ij}}{dt^2} = 2K_{ij} - \mathcal{T}_{ij} + \frac{1}{2}\int_A (x_j T_{ik} + x_i T_{jk}) n_k dS,$$

where the integral is taken over a bounding surface with the components of the outward unit normal being n_k for $k = 1, 2, 3$. On the surface and in exterior region the density is negligible,

$$I_{ij} = \int x_i x_j \rho dV, \quad K_{ij} = \int \frac{1}{2} u_i u_j \rho dV \quad \text{and} \quad \mathcal{T}_{ij} = \int T_{ij} dV$$

with the integrals being taken over the interior volume.

The initial magnetic field $\mathbf{B}_0$ is uniform and the boundary condition at large distances is that $\mathbf{B}$ is ultimately maintained at its initial value with all other quantities vanishing rapidly enough that they do not produce contributions to the surface integral. Show that as the bounding surface approaches infinity,

$$\frac{1}{2}\int_A (x_j T_{ik} + x_i T_{jk}) n_k dS - \int M_{ij} dV \to \int (M_{ij,0} - M_{ij}) dV,$$

where $M_{ij,0}$ is the initial value of M_{ij}. Hence derive the scalar virial theorem in the form

$$\frac{1}{2}\frac{d^2\mathcal{I}}{dt^2} = 2K + \Pi + W + \int \frac{1}{2\mu_0}\left(|\mathbf{B}|^2 - |\mathbf{B}_0|^2\right) dV,$$

where $\Pi = 3\int p dV$, $\mathcal{I} = I_{ii}$, and K and W are the total kinetic energy and the total gravitational energy respectively.

The gas is cold and initially at rest. Show that it will begin to collapse under gravity. Give an estimate for the change in magnetic energy that needs to occur in order to inhibit the collapse.

2

A steady state axisymmetric magnetohydrodynamic wind is such that the magnetic field may be written in the form

$$\mathbf{B} = (B_R, B_\phi, B_z) = -\frac{1}{R}\mathbf{e}_\phi \times \nabla\psi + B_\phi \mathbf{e}_\phi,$$

where ψ is the magnetic flux function and $\mathbf{e}_\phi$ is the unit vector in the azimuthal direction for cylindrical coordinates (R, ϕ, z).

Show from the continuity and induction equations that velocity field may be written in the form

$$\mathbf{u} = \frac{k\mathbf{B}}{\rho} + R\omega\mathbf{e}_\phi,$$

where $k(\psi)$ and $\omega(\psi)$ are arbitrary functions of ψ alone.

The azimuthal component of the equation of motion is

$$\rho\left(\mathbf{u}\cdot\nabla u_\phi + \frac{u_R u_\phi}{R}\right) = \frac{1}{\mu_0}\left(\mathbf{B}\cdot\nabla B_\phi + \frac{B_R B_\phi}{R}\right)$$

Use this to show that

$$Ru_\phi = \frac{RB_\phi}{\mu_0 k} + \ell,$$

where the arbitrary function $\ell = \ell(\psi)$ depends only on ψ.

Use the above relations to show that

$$u_\phi = \frac{R^2\omega - A^2\ell}{R(1 - A^2)},$$

where

$$A^2 = \frac{\mu_0 \rho u_p^2}{B_p^2},$$

where u_p and B_p are the magnitudes of the poloidal components of $\mathbf{u}$ and $\mathbf{B}$ respectively. Explain what happens when $A = 1$, stating any conditions that need to be satisfied. Hence give an expression for the specific angular momentum carried to infinity by the wind along an individual field line. Why can magnetohydrodynamic winds be efficient extractors of angular momentum?

3

A spherically symmetric supernova explosion of energy E occurs at time $t = 0$ in an ideal non magnetic gas with constant specific heat ratio γ. The undisturbed medium has density $\rho_0 = Dr^{-\beta}$ where D and β are constants. The shock is located at $r = R(t)$, and the shock speed is $\dot{R}$. The strong shock conditions relating the state variables at $r = R$ to undisturbed values may be assumed to be

$$\rho = \left(\frac{\gamma+1}{\gamma-1}\right)\rho_0, \;\; u = \frac{2\dot{R}}{\gamma+1}, \;\text{ and } p = \frac{2\rho_0 \dot{R}^2}{\gamma+1}.$$

A similarity solution for $r < R$ is sought in terms of dimensionless similarity variable $\xi = r/R(t)$. The solution has the form

$$\rho = DR^{-\beta}\tilde{\rho}(\xi), \;\; u = \dot{R}\,\tilde{u}(\xi), \;\text{ and } p = DR^{-\beta}\dot{R}^2\,\tilde{p}(\xi),$$

where $\tilde{\rho}(\xi)$, $\tilde{u}(\xi)$ and $\tilde{p}(\xi)$ are dimensionless functions to be determined. Show that for this solution the total energy of the explosion is

$$E = 4\pi\int_0^R \left(\frac{1}{2}\rho u^2 + \frac{p}{\gamma-1}\right) r^2 dr = 4\pi R^{3-\beta}\dot{R}^2 D\int_0^1 \left(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{\tilde{p}}{\gamma-1}\right)\xi^2 d\xi.$$

Hence deduce that $\dot{R}/R = 2/[(5-\beta)t]$ and that the shock front expands according to $R \propto (Et^2/D)^{1/(5-\beta)}$.

By substituting the similarity solution into the equations of ideal gas dynamics assuming spherical symmetry, show that the dimensionless functions satisfy the equations

$$(\tilde{u}-\xi)\tilde{\rho}' - \beta\tilde{\rho} = -\tilde{\rho}\tilde{u}' - \frac{2\tilde{\rho}\tilde{u}}{\xi},$$

$$(\tilde{u}-\xi)\tilde{u}' - \frac{(3-\beta)}{2}\tilde{u} = -\frac{\tilde{p}'}{\tilde{\rho}},$$

$$(\tilde{u}-\xi)\left(\frac{\tilde{p}'}{\tilde{p}} - \frac{\gamma\tilde{\rho}'}{\tilde{\rho}}\right) - 3 + \gamma\beta = 0.$$

Show that these equations have a solution for which $\tilde{u} \propto \xi$, $\tilde{\rho} \propto \xi$ and $\tilde{p} \propto \xi^3$ provided that $\beta = (7-\gamma)/(\gamma+1)$.

4

An ideally conducting star immersed in a vacuum (zero current) magnetic field undergoes linear perturbations about a spherically symmetric hydrostatic equilibrium state. The Cowling approximation in which perturbations to the gravitational potential are neglected is adopted. By linearizing the equations of ideal magnetohydrodynamics, show that the spatial dependence of a small displacement $\boldsymbol{\xi}(\mathbf{x})\exp(i\omega t)$ of the fluid is governed by the equation

$$-\omega^2\rho\boldsymbol{\xi} = \rho\mathcal{F}\boldsymbol{\xi} = \frac{\delta\rho}{\rho}\nabla p - \nabla\delta p + \frac{1}{\mu_0}(\nabla\times\delta\mathbf{B})\times\mathbf{B},$$

where ω is the oscillation frequency,

$$\delta\rho = -\nabla\cdot(\rho\boldsymbol{\xi}), \qquad \delta p = -\boldsymbol{\xi}\cdot\nabla p - \gamma p\nabla\cdot\boldsymbol{\xi} \quad \text{and} \quad \delta\mathbf{B} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}).$$

Show that the force operator $\mathcal{F}$ is self-adjoint with respect to the inner product

$$\langle\boldsymbol{\eta},\boldsymbol{\xi}\rangle = \int \rho\boldsymbol{\eta}^*\cdot\boldsymbol{\xi}dV.$$

Here the integral is taken over all space with the star assumed to be embedded in a perfectly conducting medium with negligible density under the outer boundary condition that $\delta\mathbf{B}$ and the corresponding vector potential perturbation vanish at infinity. Hence describe, stating relevant criteria, how the expression

$$\omega^2\int\rho|\boldsymbol{\xi}|^2dV = -\langle\boldsymbol{\xi},\mathcal{F}\boldsymbol{\xi}\rangle = \int\left(\frac{|\delta p|^2}{\gamma p} + \rho N^2|\xi_r|^2\right)dV + \frac{1}{\mu_0}\int|\nabla\times(\boldsymbol{\xi}\times\mathbf{B})|^2dV,$$

where $N(r)$ is given by

$$N^2 = -\frac{1}{\rho}\frac{dp}{dr}\left(\frac{1}{\gamma}\frac{d\ln p}{dr} - \frac{d\ln\rho}{dr}\right),$$

can be used, with the help of appropriate trial functions, to determine whether the system is stable or unstable. Hence show that if $N^2 \geqslant 0$ everywhere in the star it will be stable.

END OF PAPER