UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 8 June, 2011 1:30 pm to 4:30 pm

PAPER 58

STRUCTURE AND EVOLUTION OF STARS

*Attempt no more than **THREE** questions.*
*There are **FOUR** questions in total.*
The questions carry equal weight.

You may use the equations and results given below without proof.

The symbols used in these equations have the meanings that were given in lectures. Candidates are reminded of the equations of stellar structure in the form:

$$\frac{dm}{dr} = 4\pi r^2 \rho\,, \qquad \frac{dP}{dr} = -\frac{Gm\rho}{r^2}\,, \qquad \frac{dL_{\mathrm{r}}}{dr} = 4\pi r^2 \rho\epsilon\,.$$

In a radiative region

$$\frac{dT}{dr} = -\frac{3\kappa\rho L_{\mathrm{r}}}{16\pi a c r^2 T^3}\,.$$

In a convective region

$$\frac{dT}{dr} = \frac{(\Gamma_2 - 1)T}{\Gamma_2 P}\frac{dP}{dr}\,.$$

The luminosity, radius and effective temperature are related by $L = 4\pi R^2 \sigma T_{\mathrm{e}}^4\,.$

The equation of state for an ideal gas and radiation is $P = \dfrac{\mathcal{R}\rho T}{\mu} + \dfrac{aT^4}{3}\,,$ *with* $1/\mu = 2X + 3Y/4 + Z/2\,.$

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider a cluster of chemically homogeneous massive stars. The stellar material behaves as a fully ionized ideal gas with mean molecular weight μ and radiation pressure is negligible. The energy generation, opacity and whether the stellar interiors are convective or radiative depends on a star's mass. For the most massive stars energy generation is by the CNO cycle and opacity dominated by electron-scattering, such that $\epsilon = \epsilon_0 X \rho T^{13}$ and $\kappa = \kappa_0(1+X)$. Here κ_0 and ϵ_0 are constants. Use homology to show that, for such stars,

$$L \propto X\mu^{13}\frac{M^{15}}{R^{16}}.$$

At low masses they can be assumed to be fully radiative. However as the mass of a star increases convection becomes more important so that at higher masses they can be assumed to be fully convective except for a thin radiative atmosphere. Show that for radiative stars

$$L \propto \frac{\mu^4}{(1+X)}M^3$$

and for convective stars

$$L \propto \frac{\mu^4}{(1+X)^{\frac{8}{5}}}M^{\frac{12}{5}}R^{\frac{6}{5}}.$$

For zero metallicity stars with $X = 1$ determine the gradient that each zero-age main sequence has in the theoretical Hertzsprung-Russell diagram and sketch this diagram showing the two main-sequences and their transition.

Use your results to estimate how the radiative stars evolve from the zero-age main sequence. Assume they are fully mixed as hydrogen burns to helium and find the gradients $\frac{d\log L}{dX}$ and $\frac{d\log T_{\rm e}}{dX}$. Sketch an example track on your HR diagram. Comment on your evolution tracks.

2

Estimate the mean kinetic energy $< E >$ for a proton in the centre of the Sun and compare it with the Coulomb energy $E_{\rm C}$ owing to the electrostatic repulsion that must be overcome in bringing two protons together.

State briefly the two physical ideas that allow this barrier to be surmounted.

Show that in a collision between two protons, each of mass $m_{\rm p}$, the kinetic energy E in the centre of mass frame is related to their relative velocity v by $E = \frac{1}{4} m_{\rm p} v^2$.

The cross-section for nuclear reactions between two protons can be written in the form

$$\sigma(E) = \frac{S_0}{E} \exp\left(- 2\sqrt{\frac{E_{\rm B}}{E}} \right),$$

explain briefly the physical meaning of S_0 and $E_{\rm B}$.

For non-degenerate, non-relativistic gas at temperature T the relative velocity distribution is Maxwellian given by

$$n(v)dv = 4\pi (\frac{m_{\rm p}}{4\pi kT})^{\frac{3}{2}} \exp\left(- \frac{E}{kT}\right) v^2 dv.$$

The number density of reacting particles is N. Show that the reaction rate $R_{\rm pp}$ per unit volume per unit time is

$$R_{\rm pp} = \frac{1}{2} N^2 \int_0^\infty v\sigma(v) n(v) dv.$$

Deduce that

$$R_{\rm pp} = \frac{S_0 N^2}{(kT)^{3/2}} \left(\frac{4}{\pi m_{\rm p}} \right)^{\frac{1}{2}} \int_0^\infty \exp\left(- \frac{E}{kT} - 2\sqrt{\frac{E_{\rm B}}{E}} \right) dE.$$

Find the Gamow energy $E_{\rm G}$ at which the integrand peaks.

Approximate the integrand by a Gaussian centred on $E_{\rm G}$ and deduce that the temperature dependence of the reaction rates take approximately the form

$$R_{\rm pp} \propto \frac{1}{T^\alpha} \exp\left(- (\beta/T)^{\frac{1}{3}} \right),$$

where α and β are constants which you should determine. Taking logarithms and differentiating approximate the reaction rate in the form of $R_{\rm pp} \propto T^\nu$. Esimate ν for the centre of the Sun to one significant figure.

[*The temperature at the centre of the Sun* $T_{\rm c} = 2 \times 10^7 {\rm K}$, *Boltzman'sconstant* $k = 1.4 \times 10^{-16} {\rm erg\, K^{-1}}$, *the electrostatic force between two protons is* e^2/r^2 *where* $e^2 = 2.3 \times 10^{-19} {\rm erg\, cm}$, *with* $E_{\rm B} = 2 \times 10^{-7} {\rm erg}$ *and the radius of a proton,* $r_{\rm p} = 10^{-13} {\rm cm}$.]

3

Derive Schwarzschild's condition for stability to convection of a stellar radiative region consisting of an ideal gas with ratio of specific heats $\gamma = 5/3$ in the form

$$\frac{P}{T}\frac{dT}{dP} < \frac{2}{5}.$$

The temperature in the atmosphere of a cool star is given as a function of the optical depth τ by

$$T^4 = T_e^4\Big(\frac{1}{2} + \frac{3}{4}\tau\Big),$$

and the opacity is given by $\kappa = \kappa_0 \rho T^{\beta+1}$ where κ_0 and β are constants. Show that in a thin radiative layer at the surface

$$P^2 = \frac{1}{4-\beta}\frac{32 g\mathcal{R}}{3\kappa_0 \mu T_e^{\beta}}\left(\Big(\frac{1}{2} + \frac{3}{4}\tau\Big)^{\frac{4-\beta}{4}} - \Big(\frac{1}{2}\Big)^{\frac{4-\beta}{4}}\right),$$

where g is the acceleration due to gravity at the stellar surface. Deduce that convection sets in when

$$\tau = \frac{2}{3}\Big(\big(\frac{5}{1+\beta}\big)^{\frac{4}{4-\beta}} - 1\Big).$$

In the convective region just below the radius of onset the structure remains polytropic with $P = KT^{5/2}$. Show that when $\beta = 9$,

$$K^2 = \frac{32\sqrt{2}g\mathcal{R}}{15\kappa_0 \mu T_{\rm e}^{14}}.$$

In the deeper convective layers the opacity is given by $\kappa = \kappa_1 \rho T^{-3}$ where κ_1 is a constant. These layers have negligible mass, no energy generation and there is a transition to an inner radiative zone. Show that this occurs where

$$T^3 = \sqrt{2}\frac{\kappa_1}{\kappa_0}T_{\rm e}^{-10}.$$

[*You may assume that the same polytropic relation applied throught the convection zone.*]

4

Write brief notes on **four** of the following topics:

(i) The late time lightcurves of supernovae and their relation to radioactive decay.

(ii) Polytropes and their use to model white dwarfs.

(iii) The evolution of a $5M_{\odot}$ star up to second dredge-up.

(iv) The evolution of Wolf-Rayet stars.

(v) Sources of opacity over the range 10^5 to 10^8K.

END OF PAPER