UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 13 June, 2011 9:00 am to 12:00 pm

PAPER 57

STELLAR AND PLANETARY MAGNETIC FIELDS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider an $\alpha - \Omega$ dynamo model in a cartesian geometry, with mean magnetic fields that do not depend on the y-coordinate. Suppose also that the velocity is in the y direction with magnitude $V(z)$ and that the α-effect is also a function only of z. It may be assumed that α, dV/dz and the magnetic field all vanish as $z \to \pm\infty$.

Justify the following equations for the field $\mathbf{B} = B\hat{\mathbf{y}} + \nabla \times (A\hat{\mathbf{y}})$:

$$\frac{\partial A}{\partial t} = \alpha(z)B + \eta\nabla^2 A$$

$$\frac{\partial B}{\partial t} = \frac{dV}{dz}\frac{\partial A}{\partial x} + \eta\nabla^2 B$$

(i) Seek solutions of the form $A, B = \mathrm{Re}(\tilde{A}, \tilde{B})e^{ikx}$. Show that if $\int_{-\infty}^{\infty} |\tilde{A}|^2 dz$, $\int_{-\infty}^{\infty} |\tilde{B}|^2 dz$ are to increase with time, then

$$D = \frac{\max\{|\alpha|\}\max\{|dV/dz|\}}{k^3} > 1\,.$$

(ii) Now consider the case $\alpha = \alpha_0$, $dV/dz = 0$ in $z > 0$ and $\alpha = 0$, $dV/dz = \Omega_0$ in $z < 0$, where α_0, Ω_0 are constants. Seek solutions of the form

$$\begin{aligned} z > 0: \quad & \tilde{B} = P_+ e^{-\Lambda z - i\omega t}, \ \ \tilde{A} = (Q_+ + zR_+)e^{-\Lambda z - i\omega t} \\ z < 0: \quad & \tilde{A} = P_- e^{+\Lambda z - i\omega t}, \ \ \tilde{B} = (Q_- + zR_-)e^{+\Lambda z - i\omega t} \end{aligned}$$

where $\Lambda^2 = k^2 - i\omega/\eta$, and by applying appropriate jump conditions at $z = 0$ show that there is a solution with ω real when $D = 32$.

2

The linearized equations for magnetostrophic convection (in a rotating system with an imposed uniform horizontal magnetic field) in appropriate dimensionless units take the form

$$\hat{\mathbf{z}} \times \mathbf{u} = -\nabla p + \Lambda \frac{\partial \mathbf{b}}{\partial x} + qR\theta\hat{\mathbf{z}},$$
$$\frac{\partial \mathbf{b}}{\partial t} = \frac{\partial \mathbf{u}}{\partial x} + \nabla^2 \mathbf{b},$$
$$\frac{\partial \theta}{\partial t} = \mathbf{u} \cdot \hat{\mathbf{z}} + q\nabla^2 \theta,$$
$$\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{b} = 0,$$

where $\mathbf{u}, \mathbf{b}, \theta$ are the velocity, perturbation magnetic field and perturbation temperature respectively.

Explain the significance of the dimensionless numbers R, Λ, q.

Solutions are to be found for z between 0 and 1. The boundary conditions applied to the variables are that $\mathbf{u} \cdot \hat{\mathbf{z}} = \mathbf{b} \cdot \hat{\mathbf{z}} = \partial_z(\mathbf{u} \times \hat{\mathbf{z}}) = \partial_z(\mathbf{b} \times \hat{\mathbf{z}}) = \theta = 0$. Solutions can then be found in the form

$$\mathbf{u} \cdot \hat{\mathbf{z}},\ \mathbf{b} \cdot \hat{\mathbf{z}},\ \theta \ \propto\ \sin \pi z\ e^{ilx+imy+st}$$
$$p,\ \mathbf{u} \times \hat{\mathbf{z}},\ \mathbf{b} \times \hat{\mathbf{z}} \ \propto\ \cos \pi z\ e^{ilx+imy+st}.$$

(i) Derive an expression for the growth rate s in terms of the other parameters.

(ii) Find the minimum value of R as l, m and Λ are varied for steady solutions ($s = 0$). Show that this minimum value, $3\sqrt{3}\pi^2$, can be attained if $\Lambda \geqslant \sqrt{3}/2$ and give expressions for l, m in that case.

(iii) Derive an expression for R in terms of l, m, Λ in the case of oscillatory solutions ($s = i\omega$, ω real), when such solutions exist (no minimization is required).

3

The region $z > 0$ (z being measured upwards) is at rest and is permeated by a uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{x}}$. The region $z < 0$ has no magnetic field but moves at a uniform velocity $U\hat{\mathbf{x}}$ relative to the upper region. The magnetic diffusivity and viscosity of the fluid may be neglected, and the system may be regarded as isothermal.

Determine the stability of the system to small disturbances proportional to $e^{ilx+imy}$. It may be assumed that the vertical scale of the modes is much less than any scale heights of temperature and pressure so that the fluid density may be taken as constant on either side of the interface.

4

Write an essay on mean-field electrodynamics (the α-effect). Your essay should cover the α and β-effects, symmetry considerations, analytic formulae in special circumstances, nonlinear effects, spherical models and problems with interpretation at large Rm.

END OF PAPER