UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 13 June, 2011 9:00 am to 12:00 pm

PAPER 56

APPLICATIONS OF DIFFERENTIAL GEOMETRY TO PHYSICS

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Write down the Cartan-Maurer relations for the left-invariant one-forms of a Lie group. By taking an exterior derivative, show that consistency requires that the structure constants satisfy the Jacobi identity.

The vector fields

$$L_{ij} = -L_{ji} = x_i \partial_j - x_j \partial_i \tag{1}$$

generate the standard (left) action of $SO(n)$ on $\mathbb{R}^n$, where x_i, $i = 1, 2, \ldots, n$ are Cartesian coordinates.

By calculating the Lie Brackets

$$\Big[L_{ij}, L_{mn} \Big] = -C_{ij}{}^{pq}{}_{mn} L_{pq} \,, \tag{2}$$

obtain the structure constants $C_{ij}{}^{pq}{}_{mn}$ of the Lie algebra $\mathfrak{so}(n)$. Hence give the Maurer-Cartan relation for the exterior derivative of the left-invariant one-forms. Check the consistency of your result by taking an exterior derivative. Check that your general result reduces to the usual one for the case $n = 3$.

2

On a general curved four-dimensional spacetime with coordinates (t, x^i), $i = 1, 2, 3$ one may define 3-vectors $(\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H})$ with components E_i, B_i and D_i, H_i in terms of the Maxwell two-form F and its Hodge dual $\star F$ by

$$F = -E_i dt \wedge dx^i + \tfrac{1}{2} B_k \epsilon_{kij} dx^i \wedge dx^j \,, \quad \star F = H_i dt \wedge dx^i + \tfrac{1}{2} D_k \epsilon_{kij} dx^i \wedge dx^j \qquad (1)$$

where $\epsilon_{ijk} = 1$ if ijk is an even permutation of $1, 2, 3$, $\epsilon_{ijk} = -1$ if ijk is an odd permutation of $1, 2, 3$, and vanishes otherwise.

By taking an exterior derivative, obtain the four source-free Maxwell equations, in terms of $(\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H})$ and show that they are identical to their flat space forms.

If the spacetime metric is static and takes the form

$$ds^2 = -dt^2 + g_{ij}(x^k) dx^i dx^j \,, \qquad (2)$$

where g_{ij} does not depend upon t, show that the "constitutive relations"

$$D_i = \epsilon_{ij} E_j \,, \qquad B_i = \mu_{ij} H_j \qquad (3)$$

hold, where $\epsilon_{ij} = \epsilon_{ji}$ and $\mu_{ij} = \mu_{ji}$ are constructed from the determinant of g_{ij} and its inverse g^{ij}.

Give expressions for ϵ_{ij} and μ_{ij} and hence show that

$$\epsilon_{ij} = \mu_{ij} \,. \qquad (4)$$

3

An isotropic simple harmonic oscillator in three dimensions has the Hamiltonian

$$H = \tfrac{1}{2} \sum_{i=1}^{i=3} \Big(p_i p_i + q_i q_i \Big) \,. \qquad (1)$$

- Show, by introducing three complex coordinates or otherwise, that the symplectic form and and Hamiltonian are in invariant under an action of $U(3)$.
- Obtain the moment maps for all of of the generators of the $U(3)$ action.
- By writing down the equations of motion in complex form or otherwise, identify which element of the Lie algebra $\mathfrak{u}(3)$ corresponds to the Hamiltonian.
- State which moment maps are linear in p_i and which contain terms quadratic in p_i. Give the physical interpretation of the former.

4

The action of eleven-dimensional supergravity contains the term

$$S = \tfrac{1}{2} \int \big(F \wedge \star F + k A \wedge F \wedge F \big) \tag{1}$$

where F is a four-form and A a three-form such that $F = dA$, and k is a constant.

Obtain the equation of motion for F and explain why it, but not the action S , is invariant under the gauge transformation $A \to A + d\Lambda$.

Obtain the canonical energy momentum tensor for the action S.

5

Define a (left) fibre bundle and a principal bundle. Show that every principal bundle admits a global right action of the structural group. Show further that a principal bundle is trivial if and only if it admits a global section.

Define an associated bundle and illustrate your answers by reference to the oth-onormal frame bundle and the tangent bundle of a manifold. Hence show that the frame bundle and tangent bundle of a Lie group are trivial.

Give a group theoretic characterization of the orthonormal frame bundles of Minkowski spacetime, De-Sitter spacetime and Anti-De-Sitter spacetime.

END OF PAPER