1

(a) A point particle of mass m has position $x^\mu(\lambda)$ and 4-momentum $p_\mu(\lambda)$ in a spacetime with metric $g_{\mu\nu}$ $(\mu,\nu = 0,1,2,3)$, where λ is an arbitrary parameter on the particle's worldline. Starting from the action

$$I[x,p,e] = \int d\lambda \left\{ \frac{dx^\mu}{d\lambda} p_\mu - \frac{1}{2} e \left(g^{\mu\nu} p_\mu p_\nu + m^2 \right) \right\} ,$$

where e is a Lagrange multiplier and $g^{\mu\nu}$ the inverse metric, obtain the particle's equations of motion. Explain briefly why the particle moves on a geodesic. Use the equations of motion to show that the quantity $Q_\xi = \xi^\mu p_\mu$ is a constant of the motion if the vector field $\xi = \xi^\mu \partial_\mu$ is a Killing vector field.

(b) Consider the metric

$$\begin{aligned} ds^2 &= -F(r)dt^2 + 2\sqrt{1-F(r)}\, dr dt + dr^2 \\ &\quad + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \end{aligned} \qquad (1)$$

where $F(r) \to 1$ as $r \to \infty$. Why are the vector fields $k = \partial/\partial t$ and $m = \partial/\partial\phi$ Killing? Using the conservation on geodesics of $Q_k = -\epsilon$ and $Q_m = h$, and assuming $\theta = \pi/2$, show that $\dot{r}^2 + V_{\text{eff}}(r) = \epsilon^2$, where V_{eff} is a function of r that you should find (and $\dot{r}$ is the derivative of r with respect to an affine parameter). Why is no generality lost by the assumption that $\theta = \pi/2$?

(c) Show that a hypersurface $r = \mu$, for constant μ is a null hypersurface of the metric (1) when $F(\mu) = 0$. Show also that if $F < 0$ for $r < \mu$ then there is no causal future-directed worldline on which a particle passing through this hypersurface could return to $r > \mu$.

State the definition of a Killing horizon, and of its surface gravity κ. Show that a null hypersurface $r = \mu$ of the metric (1) is a Killing horizon, and compute its surface gravity.

(d) By introducing a new time coordinate $T = t + p(r)$ for some function p, show that the metric (1) is equivalent to a static metric. Find the singularities of both this static metric and the metric (1) for the case

$$F = (1 - \mu/r)^2 .$$

Use your result to explain how the static metric may be analytically continued from $r > \mu$ to $r < \mu/2$.