UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 10 June, 2011 1:30 pm to 4:30 pm

PAPER 55

BLACK HOLES

*Attempt **ALL** of Section I*
*and **TWO** of the **THREE** questions from Section II.*

Section II carries twice the weight of Section I.
Questions within each section carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

(i) Write down the equation satisfied by an arbitrarily parametrized geodesic in a general spacetime, and state the condition for the parameter to be affine. What does it mean for a spacetime to be "geodesically complete"?

(ii) The Rindler metric for a 2-dimensional flat spacetime is

$$ds^2 = -\left(\kappa x\right)^2 dt^2 + dx^2\,, \qquad x > 0\,,$$

for constant $\kappa > 0$. By defining the new coordinates

$$T = x\sinh(\kappa t)\,, \qquad X = x\cosh(\kappa t)$$

show that Rindler spacetime is a region of two-dimensional Minkowski spacetime. Show also that an observer at fixed x in the Rindler spacetime has proper acceleration of magnitude $a = 1/x$.

The Unruh effect states that an observer undergoing constant proper acceleration a in a two-dimensional Minkowski spacetime appears to be in a heat-bath at temperature $T = \hbar a/(2\pi)$. State the Tolman law for thermal equilibrium in a static spacetime and verify that this law is satisfied by the temperature of static observers in Rindler spacetime.

Explain briefly the relevance of your result to the Schwarzschild black hole.

(iii) The Kerr metric in Boyer–Linquist coordinates is

$$\begin{aligned} ds^2 \; = \; & -\Sigma^{-1}\left(\Delta - a^2\sin^2\theta\right)dt^2 \\ & -2a\Sigma^{-1}\left(r^2+a^2-\Delta\right)\sin^2\theta\,dtd\phi + \;\Sigma\,\Delta^{-1}dr^2 \\ & +\Sigma\,d\theta^2 + \Sigma^{-1}\left[\left(r^2+a^2\right)^2 - \Delta\,a^2\sin^2\theta\right]\sin^2\theta d\phi^2\,, \end{aligned}$$

where

$$\Delta = r^2 - 2Mr + a^2\,, \qquad \Sigma = r^2 + a^2\cos^2\theta\,.$$

What is the significance of the parameter a? Where is the event horizon? Where is the ergoregion? Explain briefly how the existence of an ergoregion allows energy to be extracted from a black hole (Penrose process).

(iv) Raychaudhuri's equation for a null geodesic congruence is

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}^{\mu\nu}\hat{\sigma}_{\mu\nu} + \hat{\omega}^{\mu\nu}\hat{\omega}_{\mu\nu} - R_{\mu\nu}t^\mu t^\nu\,,$$

where λ is an affine parameter. Explain briefly the significance of (a) the vector t, (b) the scalar θ, (c) the symmetric tensor $\hat{\sigma}$ and (d) the antisymmetric tensor $\hat{\omega}$.

(v) A Kerr black hole of mass M and angular momentum J has "horizon area"

$$A = 8\pi\left[M^2 + \sqrt{M^4 - J^2}\right]\,.$$

A collision of two Kerr black holes, of the same mass and angular momentum, produces a Schwarzschild black hole and gravitational radiation. What is the maximum fraction of the initial mass ($2M$) that can be radiated away without violating the second law of black hole mechanics?

(vi) A quantum field $\Phi(x)$ takes the form

$$\Phi(x) = \sum_i \left[a_i u_i(x) + a_i^\dagger u_i^*(x)\right] \qquad (*)$$

in the far past, where the spacetime is Minkowski and the functions $u_i(x)$ are the positive-frequency eigenstates of the standard Minkowski time-translation Killing vector field. The operator coefficients $(a_i, a_i^\dagger)$ satisfy the commutation relations

$$[a_i, a_j] = 0\,, \qquad \left[a_i, a_j^\dagger\right] = \delta_{ij}\,.$$

In the far future, where the spacetime is again Minkowski, the quantum field again takes the form $(*)$ but with the operators a_i replaced by

$$a_i' = \sum_j \left(a_j A_{ji} + a_j^\dagger B_{ji}^*\right)\,,$$

and hence $a_i^\dagger$ by $(a_i')^\dagger$. Find the restrictions on the matrices A and B that follow from the requirement that the primed operators obey the same commutation relations as the unprimed operators. Show that if the initial state is a vacuum state then the final state will be one with particles unless $B = 0$.

SECTION II

1

(a) A point particle of mass m has position $x^\mu(\lambda)$ and 4-momentum $p_\mu(\lambda)$ in a spacetime with metric $g_{\mu\nu}$ ($\mu,\nu = 0,1,2,3$), where λ is an arbitrary parameter on the particle's worldline. Starting from the action

$$I[x,p,e] = \int d\lambda \left\{ \frac{dx^\mu}{d\lambda} p_\mu - \frac{1}{2} e \left(g^{\mu\nu} p_\mu p_\nu + m^2 \right) \right\} ,$$

where e is a Lagrange multiplier and $g^{\mu\nu}$ the inverse metric, obtain the particle's equations of motion. Explain briefly why the particle moves on a geodesic. Use the equations of motion to show that the quantity $Q_\xi = \xi^\mu p_\mu$ is a constant of the motion if the vector field $\xi = \xi^\mu \partial_\mu$ is a Killing vector field.

(b) Consider the metric

$$\begin{aligned} ds^2 &= -F(r)dt^2 + 2\sqrt{1-F(r)}\, dr dt + dr^2 \\ &\quad + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \end{aligned} \qquad (1)$$

where $F(r) \to 1$ as $r \to \infty$. Why are the vector fields $k = \partial/\partial t$ and $m = \partial/\partial\phi$ Killing? Using the conservation on geodesics of $Q_k = -\epsilon$ and $Q_m = h$, and assuming $\theta = \pi/2$, show that $\dot{r}^2 + V_{\rm eff}(r) = \epsilon^2$, where $V_{\rm eff}$ is a function of r that you should find (and $\dot{r}$ is the derivative of r with respect to an affine parameter). Why is no generality lost by the assumption that $\theta = \pi/2$?

(c) Show that a hypersurface $r = \mu$, for constant μ is a null hypersurface of the metric (1) when $F(\mu) = 0$. Show also that if $F < 0$ for $\mu < r$ then there is no causal future-directed worldline on which a particle passing through this hypersurface could return to $r < \mu$.

State the definition of a Killing horizon, and of its surface gravity κ. Show that a null hypersurface $r = \mu$ of the metric (1) is a Killing horizon, and compute its surface gravity.

(d) By introducing a new time coordinate $T = t + p(r)$ for some function p, show that the metric (1) is equivalent to a static metric. Find the singularities of both this static metric and the metric (1) for the case

$$F = (1 - \mu/r)^2 \ .$$

Use your result to explain how the static metric may be analytically continued from $r > \mu$ to $r < \mu/2$.

2

Write an essay to explain how Carter–Penrose (CP) diagrams encode the causal structure of spherically-symmetric asymptotically-flat spacetimes. You need not present any calculations but you should explain the principles involved.

You should start with a brief explanation of conformal compactification, illustrated by Minkowski spacetime, using this to explain what is meant by "asymptotically flat". You should sketch the CP diagrams for the Schwarzschild spacetime (negative and positive mass, exterior to collapsing star, and the time symmetric maximal analytic extension) using them to illustrate the concepts of a "naked singularity", "event horizon' and "Einstein–Rosen bridge".

You should then sketch CP diagrams for the Reissner–Nordstrom black holes (collapsing charged star, maximal analytic extension, extreme case) using them to illustrate the concept of a Cauchy horizon, and to explain why Cauchy horizons are typically unstable to arbitrarily small perturbations.

3

(a) State the dominant energy condition on the stress tensor $T_{\mu\nu}$ appearing in the Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$. State the zeroth law of black hole mechanics.

Given that $R_{\mu\nu}\xi^\mu\xi^\nu = 0$ on a Killing horizon $\mathcal{N}$ of a Killing vector field ξ, use the dominant energy condition and Einstein's equations to deduce that

$$j_{[\mu}\xi_{\nu]}\Big|_{\mathcal{N}} = 0\,, \qquad j_\mu = -T_{\mu\nu}\xi^\nu\,.$$

Given also that $\xi_{[\mu}R_{\nu]}{}^\lambda\xi_\lambda = -2\xi_{[\mu}\partial_{\nu]}\kappa$, where κ is the surface gravity of $\mathcal{N}$, deduce the zeroth law. [You may assume that the future event horizon of an asymptotically-flat stationary black hole spacetime is a Killing horizon.]

(b) The Komar integral

$$Q_\xi(V) = \frac{c}{16\pi G}\oint_{\partial V} dS_{\mu\nu} D^\mu \xi^\nu$$

gives (for some constant c) the "charge" in volume $V \subset \Sigma$ (a spacelike hypersurface) associated with a Killing vector field ξ. Using the fact that $D_\rho D_\mu \xi^\nu = R^\nu{}_{\mu\rho\sigma}\xi^\sigma$, show that

$$Q_\xi(V) = \int_V dS_\mu J^\mu_\xi\,,$$

for a 4-vector current J^μ_ξ, which you should find, constructed from ξ and the stress tensor $T_{\mu\nu}$. Show further that

$$D_\mu J^\mu_\xi = 0\,.$$

(c) For an asymptotically-flat spacetime with time-translation and rotation Killing vectors k and m, respectively, the Komar integrals for total mass and angular momentum are

$$M = -\frac{1}{8\pi G} \oint_\infty dS_{\mu\nu} D^\mu k^\nu \,, \qquad J = \frac{1}{16\pi G} \oint_\infty dS_{\mu\nu} D^\mu m^\nu \,.$$

Use these integrals to obtain expressions for the mass and angular momentum of a stationary, axi-symmetric, asymptotically-flat, black hole solution of the vacuum Einstein equations as integrals over a section of its event horizon. Using the zeroth law of black hole mechanics, and assuming constant angular velocity Ω_H of the horizon, derive the Smarr formula

$$M = \frac{\kappa A}{4\pi} + 2\Omega_H J \,,$$

in units for which $G = 1$, where κ is the surface gravity and A is the "horizon area". Use this formula to obtain a version of the "first law of black hole mechanics".

END OF PAPER