UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 13 June, 2011 1:30 pm to 3:30 pm

PAPER 54

ADVANCED COSMOLOGY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) In the 3+1 formalism, we split spacetime using the line element

$$ds^2 = -N^2 dt^2 + {}^{(3)}g_{ij}(dx^i - N^i dt)(dx^j - N^j dt)\,,$$

with lapse function $N(t, x^i)$, shift vector $N^i(t, x^i)$ and the three-metric ${}^{(3)}g_{ij}(t, x^i)$ on constant time spacelike hypersurfaces Σ. (Latin indices vary over 1,2,3.) The extrinsic curvature of Σ is given by $K_{\alpha\beta} = P^\mu_\alpha P^\nu_\beta \nabla_\mu n_\nu$ where the projector $P^\mu_\alpha = \delta^\mu_\alpha + n_\alpha n^\mu$ and n^μ is normal to Σ with $g_{\mu\nu} n^\mu n^\nu = -1$. The derivative operator $\mathcal{D}_i$ on Σ can be defined by projecting the 3+1 covariant derivative ∇_μ onto Σ using P^μ_i.

(i) Show that $P^\mu_\alpha P^\alpha_\nu = P^\mu_\nu$.

(ii) Prove that $\mathcal{D}_k\left({}^{(3)}g_{ij}\right) = 0$, where we note that the induced metric ${}^{(3)}g_{ij}$ can be represented as ${}^{(3)}g_{\alpha\beta} = P^\mu_\alpha\, P^\nu_\beta\, g_{\mu\nu} = g_{\alpha\beta} + n_\alpha n_\beta$.

(iii) Show also that $P^\mu{}_\lambda P^\nu{}_\sigma\, \nabla_\nu\left(P^\alpha{}_\mu\right) = K_{\lambda\sigma}\, n^\alpha$.

(b) When linearising the 3+1 metric about a flat FRW universe $ds^2 = \bar{N}^2 dt^2 - a^2 d\mathbf{x}^2$, we define the scalar perturbations by

$$N(t, x^i) \equiv \bar{N}(t)(1 + \Phi(t, x^i))\,, \qquad N_i \equiv -a^2 B_{,i}\,,$$
$${}^{(3)}g_{ij} = a^2[(1 - 2\Psi)\delta_{ij} - 2E_{,ij}]\,,$$

where bars denote background homogeneous quantities.

Under the change of coordinates

$$(t,\ x^i) \longrightarrow (\tilde{t},\ \tilde{x}^i) = (t + \xi^0,\ x^i + \xi^i)$$

(with $\xi^i \equiv \partial^i \lambda$), metric perturbations transform as

$$\delta\tilde{g}_{ij} = \delta g_{ij} - \bar{g}_{ij,0}\xi^0 - \bar{g}_{kj}\xi^k_{,i} - \bar{g}_{ik}\xi^k_{,j}\,.$$

The adiabatic perturbation is defined by

$$\zeta = -\Psi + \tfrac{1}{3}\frac{\delta\rho}{\bar{\rho} + \bar{P}}$$

where $\rho = \bar{\rho} + \delta\rho$ and $P = \bar{P} + \delta P$ are the background density and pressure respectively.

(i) Prove that ζ is gauge-invariant.

(ii) Show that ζ is independent of time in the long wavelength approximation.

(iii) Briefly discuss the advantages of using ζ to describe cosmological perturbations.

[You may assume a definite equation of state $P = w\rho$, that the perturbed energy density conservation equation is

$$\dot{\delta\rho}/\bar{N} = -3H(\delta\rho + \delta P) + (\bar{\rho} + \bar{P})(\kappa - 3H\Phi) - \triangle u\,,$$

and that the metric perturbation Ψ satisfies

$$\dot{\Psi}/\bar{N} = -H\Phi + \tfrac{1}{3}\kappa + \tfrac{1}{3}\triangle\chi\,,$$

where $\triangle \equiv \nabla^2/a^2$, u generates the scalar velocity perturbation, and κ and χ generate the trace and traceless part of K_{ij} respectively.]

2

Consider photon propagation in a perturbed FRW universe (flat $\Omega = 1$) with line element (synchronous gauge):

$$ds^2 = a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j\right] . \qquad (1)$$

The photon has four-momentum p^μ ($p_\mu p^\mu = 0$) and a comoving observer with four-velocity $u^\mu = a^{-1}(1,\, 0,\, 0,\, 0)$ measures the photon energy to be $E = -u_\mu p^\mu = ap^0 \equiv q/a$ where q is the comoving momentum. The comoving wavevector $\mathbf{k}$ has wavenumber $k = |\mathbf{k}|$ and direction $\hat{k}^i = k^i/k$.

(i) Using the geodesic equation $\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\nu\sigma}p^\nu p^\sigma = 0$, show that a photon propagating along a direction $\hat{\mathbf{n}}$ will have a trajectory that satisfies the following to linear order:

$$\frac{dq}{d\tau} = -\frac{1}{2}qh'_{ij}\hat{n}^i\hat{n}^j\,, \qquad\qquad \frac{d\hat{n}^i}{d\tau} = \mathcal{O}(h_{ij})\,.$$

Briefly discuss the significance of these results for solving the Einstein–Boltzmann equations at linear order.

[You may assume that the connection to linear order for the metric (1) is given by $\Gamma^0_{00} = \frac{a'}{a}$, $\Gamma^0_{0i} = 0$, $\Gamma^0_{ij} = \frac{a'}{a}(\delta_{ij} + h_{ij}) + \frac{1}{2}h'_{ij}$, $\Gamma^i_{0j} = \frac{a'}{a}\delta_{ij} + \frac{1}{2}h'_{ij}$ and $\Gamma^i_{jk} = \frac{1}{2}(h_{ij,k} + h_{ik,j} - h_{jk,i})$.]

(ii) Assume that the photon brightness function $\Delta(x^i,\, \hat{n}^i,\, \tau) \equiv 4\Delta T/T$ satisfies the collisionless Boltzmann equation which in Fourier space is given by

$$\Delta' + ik\mu\Delta = -2h'_{ij}\hat{n}^i\hat{n}^j\,, \qquad (2)$$

where $\mu = \hat{\mathbf{k}}\cdot\hat{\mathbf{n}}$. If the photon fluid is in equilibrium prior to decoupling $\tau \leqslant \tau_{\rm dec}$, we can approximate its initial conditions at decoupling ($\tau \approx \tau_{\rm dec}$) by

$$\Delta(\mathbf{k},\, \mu,\, \tau_{\rm dec}) = \delta_\gamma(\tau_{\rm dec}) + 4\mathbf{n}\cdot\mathbf{v}(\tau_{\rm dec})\,,$$

where δ_γ and $\mathbf{v}$ are the photon density and velocity fluctuations.

Assuming instantaneous decoupling at $\tau = \tau_{\rm dec}$, integrate (2) from decoupling to the present day $\tau = \tau_0$ to find the Sachs–Wolfe formula for the CMB temperature anisotropy seen at position $\mathbf{x}$ in a direction $\hat{\mathbf{n}}$:

$$\frac{\Delta T}{T}(\mathbf{x},\mathbf{n},\tau_0) = \frac{1}{4}\delta_\gamma(\tau_{\rm dec}) + \mathbf{n}\cdot\mathbf{v}(\tau_{\rm dec}) - \frac{1}{2}\int_{\tau_{\rm dec}}^{\tau_0} d\tau h'_{ij}\ \hat{n}^i\hat{n}^j\,. \qquad (3)$$

Explain the meaning of each term in the formula (3), and specify the angular scales on which these contributions are important. Sketch a typical angular power spectrum for $\Delta T/T$ to illustrate these contributions.

3

Consider the following term in the interaction Hamiltonian for a non-canonical theory of inflation

$$H_{int}(\tau) = \int d^3x \, \frac{\epsilon}{c_s^4}(\epsilon - 3 + 3c_s^2) a(\tau)\zeta(\mathbf{x},\tau)(\zeta'(\mathbf{x},\tau))^2 \,,$$

where primes denote derivatives with respect to conformal time τ, i.e. $d/dt = a^{-1}d/d\tau$. The slow-roll parameter ϵ and the sound speed $c_s^2 \leqslant 1$ are varying very slowly with time, so for the purpose of this calculation you can assume that they are constant in time.

During inflation, we can expand the *interaction picture* field ζ_I in the following way

$$\zeta_I(\mathbf{x},\tau) = \int \frac{d^3k}{(2\pi)^3}\left[a_I(\mathbf{k})u_k^*(\tau)e^{i\mathbf{k}\cdot\mathbf{x}} + a_I^\dagger(\mathbf{k})u_k(\tau)e^{-i\mathbf{k}\cdot\mathbf{x}}\right] = \zeta_I^+(\mathbf{x},\tau) + \zeta_I^-(\mathbf{x},\tau)\,,$$

where the mode function has the following solution

$$u_k(\tau) = \frac{H}{\sqrt{4\epsilon c_s k^3}}(1 - ikc_s\tau)e^{ic_sk\tau}.$$

(i) Using this interaction Hamiltonian, show that the 3-point correlation function at $\tau \to 0$

$$\begin{aligned} &\langle\zeta(\mathbf{k}_1,\tau)\zeta(\mathbf{k}_2,\tau)\zeta(\mathbf{k}_3,\tau)\rangle \\ &\quad = \mathrm{Re}\left\langle\left[-2i\zeta_I(\mathbf{k}_1,\tau)\zeta_I(\mathbf{k}_2,\tau)\zeta_I(\mathbf{k}_3,\tau)\int_{-\infty(1+i\epsilon)}^{\tau} d\tau'\, a(\tau')H_{int}^I(\tau')\right]\right\rangle \end{aligned} \qquad (1)$$

is given by

$$\begin{aligned} &\langle\zeta(\mathbf{k}_1,0)\zeta(\mathbf{k}_2,0)\zeta(\mathbf{k}_3,0)\rangle = \\ &\frac{\epsilon - 3 + 3c_s^2}{\epsilon^2 c_s^4}\frac{H^4}{16}\frac{1}{(k_1k_2k_3)^3}(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)(k_2k_3)^2\left(\frac{1}{K}+\frac{k_1}{K^2}\right) + 1\to 2 + 1 \to 3. \end{aligned}$$

[You may assume that the scale factor $a(\tau) = -1/(H\tau)$ and τ runs from $-\infty < \tau < 0$.]

(ii) Write down the contribution to the 3-point correlation function for this interaction term in the following two limits, assuming that $\epsilon \approx 0.01$,

- $c_s^2 \to 1$,
- $c_s^2 \ll 1$.

What is the ratio of non-Gaussianity generated by the above two terms? Compare and comment on their relative magnitude as a function of c_s^2.

4

(a) Consider the following Lagrange density up to 3rd order in perturbation theory

$$\mathcal{L} = \frac{1}{2}\dot{\zeta}^2 - \frac{1}{2}(\partial\zeta)^2 + \frac{1}{2}m_\zeta^2\zeta^2 + \alpha\dot{\zeta}^3 + \beta\zeta(\partial\zeta)^2 + \gamma\zeta(\dot{\zeta})^2 \,. \tag{1}$$

Calculate the canonical momentum π

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\zeta}} \,. \tag{2}$$

Hence, calculate the Hamiltonian density for this action $\mathcal{H}(\pi,\zeta)$ to *third* order in perturbation theory. Identify the *interaction Hamiltonian density* $\mathcal{H}_{int}$.

(b) The optimal estimator for stochastic gravity waves detection is given by

$$\mathrm{SNR}^2 = 2T\int_0^\infty df\, \frac{S_h(f)^2\Gamma(f)^2}{N^2(f)} \,, \tag{3}$$

where $\Gamma(f)$ is the *overlap reduction function*, T is the total integration time of the experiment, and the noise spectral density of the experiment can be approximated by the tophat function

$$N(f) = \begin{cases} 10^{-44}\ \mathrm{Hz}^{-1} \,, & 10\ \mathrm{Hz} < f < 100\ \mathrm{Hz} \,, \\ \gg 1\ \mathrm{Hz}^{-1} \,, & \text{otherwise.} \end{cases} \tag{4}$$

The signal spectral density is given by

$$S_h(f) = \frac{3H_0^2}{4\pi^2}\frac{1}{f^3}\Omega_{gw}(f). \tag{5}$$

You can assume that $\Gamma(f) = 1$ for the following calculation.

(i) Inflation predicts a scale invariant $\Omega_{gw}(f)$ which is independent of f, and current CMB polarization data constrain it to be $< 10^{-14}$. Assume that the current Hubble constant is $H_0 = 100$km/s/Mpc, and each parsec 1 pc = 3.26 light years, *estimate* the lower bound on the total integration time T in *years* required for a detection (i.e. SNR> 1).

(ii) Given that for a total integration time of 5 years no detection has been made, what is the upper bound on a scale-invariant $\Omega_{gw}(f)$ given this detector?

[It is sufficient to make order of magnitude estimates. Note that the speed of light is $c = 3 \times 10^{10}$ cm/s and that 1 Hz = 1 s^{-1}.]

END OF PAPER