UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 8 June, 2011 9:00 am to 12:00 pm

PAPER 53

COSMOLOGY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The evolution of a homogeneous and isotropic universe with expansion scale factor $a(t)$ is described by solutions of the Friedmann equations:

$$\begin{aligned} \frac{\dot{a}^2}{a^2} &= \frac{8\pi G\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3} , \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} , \end{aligned}$$

where k is the curvature constant, Λ is the cosmological constant, and ρ and p are the energy density and pressure of the material content of the universe; overdots denote differentiation with respect to the comoving proper time, t.

Define the Hubble expansion rate, H, and the deceleration parameter, q, of the universe.

If the universe contains only pressureless 'dust' and a cosmological constant, show that

$$K = 4\pi G\rho - H^2(q+1) ,$$

where $K \equiv ka^{-2}$.

Define the 'surge' of the expansion by

$$Q \equiv \frac{\dddot{a}}{aH^3} ,$$

and show that

$$\begin{aligned} K &= H^2(Q-1) , \\ \Lambda &= H^2(Q-2q) , \\ 4\pi G\rho &= H^2(Q+q) . \end{aligned}$$

Define the density parameters Ω_k, Ω_m and Ω_Λ and express each of them in terms of q and Q. What does observational evidence suggest is the approximate value of the surge, Q, today? What can you conclude from the sign of q today?

2

Massive particles and antiparticles with mass m and number densities $n(m,t)$ and $\overline{n}(m,t)$ are present at time t in the radiation era of an expanding universe with zero curvature and no cosmological constant. If they interact with cross-section σ at velocity v, explain why the evolution of $n(m,t)$ is described by

$$\frac{\partial n}{\partial t} = -3\frac{\dot{a}}{a}n - n\overline{n}\langle\sigma v\rangle + P(t) \ ,$$

where the expansion scale factor of the universe is $a(t)$, and identify the physical significance of each of the terms appearing in this equation.

By considering the evolution of the antiparticles, show that

$$(n-\overline{n})a^3 = \text{ constant.}$$

Assuming initial particle-antiparticle symmetry, show that

$$\frac{d(na^3)}{dt} = \langle\sigma v\rangle\,(n_{eq}^2 - n^2)a^3,$$

where n_{eq} denotes the equilibrium number density.

Define $Y \equiv n/T^3$ and $x \equiv m/T$, and show that

$$\frac{dY}{dx} = -\frac{\lambda}{x^2}(Y^2 - Y_{eq}^2),$$

where $\lambda = m^3\langle\sigma v\rangle/H(m)$, g is the particle spin weight, and $H(m)$ is the Hubble expansion rate at temperature $T = m$.

If the number density stays close to its equilibrium value

$$n_{eq} = g(mT/2\pi)^{3/2}\exp(-m/T) \ ,$$

in units such that $\hbar = c = k_B = 1$, show that the particle-antiparticle equilibrium is significantly broken when x grows to a freeze-out value x_f determined by

$$\langle \sigma v \rangle \, gm g_*^{-1/2} x_f^{1/2} \exp(-x_f) = C \ ,$$

where g_* is the total number of relativistic degrees of freedom in the universe and C is a constant.

When $x > x_f$, the number density, n, can be assumed to be depleted only by particle-antiparticle annihilations. If λ is constant show that at late times Y approaches a value given by

$$Y_\infty = \frac{x_f}{\lambda} \ .$$

Explain the dependence of Y_∞ on $\langle \sigma v \rangle$ and sketch the schematic evolution of Y versus x for both a strongly and a weakly interacting population of annihilating particles and antiparticles.

If there was a speed-up in the expansion rate of the universe caused by the addition of extra low-mass neutrino species what would happen to the abundance of surviving massive particles and why?

For proton and antiproton annihilation the resulting number densities of surviving protons and antiprotons relative to the photon number density, n_γ, are calculated to be

$$\frac{n}{n_\gamma} = \frac{\overline{n}}{n_\gamma} = 10^{-19}.$$

How does this compare with observational data? What do you conclude about the abundances of protons and antiprotons in the early universe?

3 Consider a universe dominated at early times by radiation and pressure-free matter. Denote by Ω_r and Ω_m their present-day density parameters. Show that the conformal Hubble parameter $\mathcal{H}$ satisfies

$$\mathcal{H}^2 = \frac{\mathcal{H}_0^2 \Omega_m^2}{\Omega_r}\left(\frac{1}{y} + \frac{1}{y^2}\right) ,$$

where $\mathcal{H}_0$ is the present value of $\mathcal{H}$ and $y \equiv a/a_{\rm eq}$ is the ratio of the scale factor to its value when the energy density of the matter and radiation are equal.

Describe qualitatively the behaviour of the conformal-Newtonian-gauge (CNG) fractional density perturbations δ_r and δ_m for a scalar perturbation at scale k, with adiabatic initial conditions, that re-enters the Hubble radius well before $a_{\rm eq}$.

For perturbations on scales much smaller than the Hubble radius, the fluctuations in the radiation can be neglected. The continuity and Euler equations for the matter, and the 00 Einstein equation are then

$$\begin{aligned} \dot{\delta}_m + \nabla^2 v_m - 3\,\dot{\phi} &= 0\,, \\ \dot{v}_m + \mathcal{H} v_m + \phi &= 0\,, \\ \nabla^2\phi - 3\mathcal{H}\left(\dot{\phi} + \mathcal{H}\phi\right) &= 4\pi G a^2 \overline{\rho}_m \delta_m\,, \end{aligned}$$

where $\partial_i v_m$ is the peculiar velocity of the matter, ϕ is the CNG metric perturbation, $\bar{\rho}_m$ is the background matter density, and dots denote derivatives with respect to conformal time. Assuming that ϕ evolves on a Hubble timescale, show that

$$\ddot{\delta}_m + \mathcal{H}\,\dot{\delta}_m - 4\pi G a^2 \overline{\rho}_m \delta_m \approx 0\,. \qquad (*)$$

Show further that, in terms of the variable y, equation $(*)$ becomes

$$\frac{d^2\delta_m}{dy^2} + \frac{2+3y}{2y(1+y)}\frac{d\delta_m}{dy} - \frac{3}{2y(1+y)}\delta_m = 0\,.$$

Hence verify that the solutions are

$$\delta_m \propto 2+3y\ ,$$

$$\delta_m \propto (2+3y)\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right) - 6\sqrt{1+y}\,.$$

Argue that δ_m grows like $\ln y$ for $y \ll 1$ but as y for $y \gg 1$.

4 Consider inflation driven by a single scalar field Φ with potential $V(\Phi)$. Given that the stress-energy tensor of a scalar field is

$$T_{\mu\nu} = \nabla_\mu \Phi \nabla_\nu \Phi - g_{\mu\nu}\left(\frac{1}{2}\nabla^\rho \Phi \nabla_\rho \Phi - V(\Phi)\right),$$

derive expressions for the energy density and pressure of a homogeneous scalar field $\overline{\Phi}(t)$ in an unperturbed flat, Robertson-Walker geometry. Hence show that the equation of motion for the field is

$$\partial_t^2 \overline{\Phi} + 3H\partial_t \overline{\Phi} + V'(\overline{\Phi}) = 0\,,$$

where H is the Hubble parameter and primes denote derivatives with respect to $\overline{\Phi}$.

What is meant by slow-roll inflation? Defining slow-roll parameters,

$$\epsilon_V \equiv \frac{M_{\rm Pl}^2}{2}\left(\frac{V'}{V}\right)^2, \qquad \eta_V \equiv M_{\rm Pl}^2 \frac{V''}{V}\,,$$

where $M_{\rm Pl}^2 = 1/(8\pi G)$ is the reduced Planck mass, calculate the value of ϵ_V at the end of inflation assuming slow-roll holds up until this point.

Given that the power spectrum of the comoving curvature perturbation $\mathcal{R}$ from slow-roll inflation is

$$\mathcal{P}_{\mathcal{R}}(k) \approx \left(\frac{H^2}{2\pi \partial_t \overline{\Phi}}\right)^2,$$

where the right-hand side is evaluated at horizon exit ($k = aH$), show that the spectral index $n_s(k) = 1 + d\ln \mathcal{P}_{\mathcal{R}}(k)/d\ln k$ is

$$n_s(k) = 1 + 2\eta_V(\overline{\Phi}) - 6\epsilon_V(\overline{\Phi})$$

to leading-order in the slow-roll parameters.

Observations indicate a power-law spectrum $\mathcal{P}_{\mathcal{R}}(k)$ with amplitude $A_s \approx 2.44 \times 10^{-9}$ at a typical cosmological scale k_0 and $1 - n_s = 0.037 \pm 0.012$. Assuming modes at k_0 exited the horizon $N = 60$ e-folds before the end of inflation, show that a potential $V = m^2\Phi^2/2$ is consistent with these observations and estimate the ratio $m/M_{\rm Pl}$.

END OF PAPER