UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 6 June, 2011 9:00 am to 12:00 pm

PAPER 52

GENERAL RELATIVITY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a)(i) Explain how the Newtonian equation of motion of a particle in a time-independent gravitational field can be obtained from the geodesic equation in a spacetime with metric

$$ds^2 = -(1+2\Phi(x,y,z))dt^2 + (1-2\Phi(x,y,z))(dx^2+dy^2+dz^2)$$

where $|\Phi| \ll 1$.

(ii) Show that the Einstein equation implies that a time-independent gravitational field produced by a weak, time-independent, non-relativistic, distribution of matter must be described by a metric of the above form, where Φ satisfies Newton's law of gravitation. (You may quote the linearized Einstein equation from Q3 below.)

(b) A certain satellite is expected to malfunction after time T. The satellite follows the innermost stable circular orbit ($r = 6M$) around a Schwarzschild black hole of mass M, with metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

How many orbits will the satellite complete before it malfunctions?

2

(a) Consider an infinitesimal variation of the spacetime metric $g_{ab} \to g_{ab} + \delta g_{ab}$.

(i) Show that the corresponding change in the Levi-Civita connection is given by

$$\delta\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\nabla_c \delta g_{db} + \nabla_b \delta g_{dc} - \nabla_d \delta g_{bc}\right)$$

where ∇ is the Levi-Civita connection associated to g_{ab}.

(ii) Show that the change in the Ricci tensor is

$$\delta R_{ab} = \nabla_c \delta\Gamma^c_{ab} - \nabla_b \delta\Gamma^c_{ac}$$

(You may use the formula for the components of the Riemann tensor in a coordinate basis:

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\tau_{\nu\sigma}\Gamma^\mu_{\tau\rho} - \Gamma^\tau_{\nu\rho}\Gamma^\mu_{\tau\sigma} \Big)$$

(iii) Show that the change in the Ricci scalar is

$$\delta R = -R^{ab}\delta g_{ab} + \nabla^a \nabla^b \delta g_{ab} - \nabla^c \nabla_c \left(g^{ab}\delta g_{ab}\right)$$

(b) Consider a theory of gravity coupled to a scalar field defined by the action

$$S = \int d^4x \sqrt{-g}\, e^\Phi \left(R + g^{ab}\nabla_a \Phi \nabla_b \Phi\right)$$

Derive the equations of motion that arise from varying the scalar field and the metric. Hence show that

$$R_{ab} = \nabla_a \nabla_b \Phi$$

(You may use without proof the formula $\delta g = g g^{ab} \delta g_{ab}$.)

3

In the study of linearized perturbations of Minkowski spacetime it is assumed that there exist global coordinates $x^\mu = (t, \mathbf{x})$ with respect to which the metric has components

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where the components of $h_{\mu\nu}$ have absolute values much smaller than 1.

(a) Diffeomorphisms are a gauge symmetry in General Relativity. By considering the effect of a 1-parameter family of diffeomorphisms ϕ_s for small s, explain why linearized theory has the gauge symmetry $h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}$ where ξ_μ is small.

(b) Let $\bar{h}_{\mu\nu} = h_{\mu\nu} - (1/2)h\,\eta_{\mu\nu}$ where $h = \eta^{\mu\nu}h_{\mu\nu}$. Explain why one can impose the gauge condition

$$\partial^\mu \bar{h}_{\mu\nu} = 0.$$

(c) The linearized Einstein equation in the above gauge is

$$\partial^\rho \partial_\rho \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$$

Consider a localized distribution of matter. Let $r^2 = \mathbf{x}^2$. Show that, for large r,

$$\bar{h}_{ij}(t, \mathbf{x}) \approx \frac{2}{r}\ddot{I}_{ij}(t - r)$$

where indices i, j refer to spatial directions, a dot denotes a derivative, and

$$I_{ij}(t) = \int d^3x\, x^i x^j T_{00}(t, \mathbf{x})$$

(d) State the quadrupole formula for the power radiated in gravitational waves. Consider a pair of particles of mass m connected by a spring of negligible mass undergoing simple harmonic motion, with positions $x = \pm d(1/2 + (1/4)\sin\omega t)$, $y = z = 0$. Determine the average power radiated in gravitational waves. (You may assume that the motion is non-relativistic so the energy density of a single particle at $\mathbf{x}(t)$ is $T_{00} = m\,\delta^3(\mathbf{x} - \mathbf{x}(t))$.)

4

(a) In global coordinates, the metric on de Sitter spacetime is

$$ds^2 = -dt^2 + L^2 \cosh^2(t/L) \left[d\chi^2 + \sin^2\chi \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

(i) Is there anything special about points with $t = 0$? Give a brief explanation of your answer.

(ii) Explain how to construct the Penrose diagram of this spacetime. What do the surfaces $\mathcal{I}^{\pm}$ in this diagram represent?

(iii) Explain the concept of a *cosmological horizon* using de Sitter spacetime as an example.

(b) What is a *particle horizon* in a FLRW universe? Explain what is meant by the *horizon problem* in cosmology.

(c) The Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2}$$

(i) Assume that the universe contains pressureless matter with energy density ρ_m, and a cosmological constant Λ. For $k = 1$, show that there is a critical value for ρ_m for which $a(t)$ is independent of time. Determine a in terms of Λ.

(ii) Is this solution stable if ρ_m is perturbed from its critical value? Explain your reasoning.

END OF PAPER