UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 8 June, 2011 1:30 pm to 3:30 pm

PAPER 51

QUANTUM FOUNDATIONS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Consider a system whose Hilbert space $\mathcal{H}$ is finite-dimensional. A mixed state of the system is defined by the ensemble of vectors $\{|\psi_i\rangle\}_{i=1}^n$ with probabilities $\{p_i\}_{i=1}^n$. Write down the corresponding density matrix ρ. Show that ρ is self-adjoint and positive semi-definite and that $\mathrm{Tr}(\rho) = 1$. Show, conversely, that if a matrix ρ is self-adjoint and positive semi-definite, and obeys $\mathrm{Tr}(\rho) = 1$, then there exists an ensemble of vectors for which ρ is the density matrix.

(b) Let $\mathcal{D}_{\mathcal{H}}$ be the set of density matrices defining mixed states of the system above. Define a convex decomposition of a density matrix ρ in $\mathcal{D}_{\mathcal{H}}$ to be an expression $\rho = \sum_i a_i \rho_i$, where each ρ_i is in $\mathcal{D}_{\mathcal{H}}$ and each $a_i > 0$, and where $\sum_i a_i = 1$. Define a density matrix ρ in $\mathcal{D}_{\mathcal{H}}$ to be *pure* if, given any convex decomposition of ρ, we have that $\rho_i = \rho$ for all i. Show that ρ is pure (by this definition) if and only if $\rho = |\psi\rangle\langle\psi|$ for some state vector $|\psi\rangle$ in $\mathcal{H}$.

(c) Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ define the Hilbert space of a combined system $S = S_1 + S_2$, and let $|\psi\rangle$ be a pure state of the combined system. Define the reduced density matrix of $|\psi\rangle$ on S_1, explaining your notation clearly. Now consider a general quantum measurement $\{A_i\}$ made on S_2 when the combined system is in state $|\psi\rangle$. Give expressions for the probability p_i of obtaining outcome i, and for the state $|\psi_i\rangle$ of the combined system after a measurement with this outcome. Hence show that the reduced density matrix on S_1 is unaltered by the measurement. Comment briefly on what this says about the relationship between quantum theory and special relativity.

2

Alice and Bob each have a box which accepts integer inputs in the range $\{0, \ldots, N-1\}$ and produces outputs 0 or 1 in response to each input. Suppose that the boxes follow deterministic classical algorithms (which need not necessarily be the same) for generating outputs from inputs, and write a_i and b_i for their respective outputs given input i. Show that at least one of the statements in the list L given by $a_0 = b_0$, $b_0 = a_1$, $a_1 = b_1$, $b_1 = a_2$, $\ldots$, $a_{N-1} = b_{N-1}$, $b_{N-1} \neq a_0$ must be false. Hence show that, if the boxes' outputs are defined by any (probabilistic or deterministic) local hidden variables theory, and Alice and Bob choose at random one of the input pairs $(0,0), (1,0), (1,1), (2,1), \ldots, (N-1, N-1), (0, N-1)$, the probability of the corresponding statement in L being false must be at least $\frac{1}{2N}$.

Give an explicit calculation to show that this is not the case if the boxes share entangled quantum states and can use measurements on these states to determine their outputs.

3

Consider a system $\mathcal{S}$ with two spin states $\{|\Uparrow\rangle, |\Downarrow\rangle\}$ that interacts with an environment $\mathcal{E}$ described by a collection of N other two-state spins represented by $\{|\uparrow_k\rangle, |\downarrow_k\rangle\}$, $k = 1 \cdots N$. Let the system and environment Hamiltonians $H_{\mathcal{S}}$ and $H_{\mathcal{E}}$ and the self-interaction Hamiltonian $H_{\mathcal{E}\mathcal{E}}$ of the environment be equal to zero. Take the interaction Hamiltonian $H_{\mathcal{S}\mathcal{E}}$ that describes the coupling of the spin of the system to the spins of the environment to be of the form

$$H_{\mathcal{S}\mathcal{E}} = (|\Uparrow\rangle\langle\Uparrow| - |\Downarrow\rangle\langle\Downarrow|) \otimes \sum_k g_k(|\uparrow_k\rangle\langle\uparrow_k| - |\downarrow_k\rangle\langle\downarrow_k|) \bigotimes_{k' \neq k} I_{k'}, \tag{1}$$

where the g_k are coupling constants and $I_k = (|\uparrow_k\rangle\langle\uparrow_k| + |\downarrow_k\rangle\langle\downarrow_k|)$ is the identity operator for the kth environmental spin. Suppose that before the interaction is turned on the system and environment are in the initial state

$$|\psi(0)\rangle = (a|\Uparrow\rangle + b|\Downarrow\rangle) \bigotimes_{k=1}^{N} (\alpha_k|\uparrow_k\rangle + \beta_k|\downarrow_k\rangle) \,. \tag{2}$$

Show that $\rho_{\mathcal{S}}(t)$, the system's reduced density matrix at time t, takes the form

$$\begin{aligned} \rho_{\mathcal{S}}(t) \quad &= \quad |a|^2|\Uparrow\rangle\langle\Uparrow| + |b|^2|\Downarrow\rangle\langle\Downarrow| \\ &\quad\quad + z(t)ab^*|\Uparrow\rangle\langle\Downarrow| + z^*(t)a^*b|\Downarrow\rangle\langle\Uparrow| \,. \end{aligned} \tag{3}$$

Give an expression for the interference coefficient $z(t)$. Show that for typical randomly chosen values of the coefficients α_k, β_k and g_k the long time average of $z(t)$ tends to zero as $N \to \infty$.

Comment briefly on the physical significance of this result.

4

Consider a model of a quantum system on a one-dimensional lattice in which the sites are labelled by integers and are either occupied by a single particle or vacant. Suppose that the system has zero Hamiltonian and that measurements of the particle occupation number spontaneously take place on randomly chosen sites at random times, with expected time interval T between measurements for any given site. Write $|i, 1\rangle$ for the state of the system $\ldots |0\rangle \otimes |0\rangle \otimes \ldots \otimes |0\rangle \otimes |1\rangle \otimes |0\rangle \otimes \ldots$, with a single particle occupying site i, and write $|i, N\rangle$ for the state $\ldots |0\rangle \otimes |0\rangle \otimes \ldots \otimes |0\rangle \otimes |1\rangle \otimes \ldots \otimes |1\rangle \otimes |0\rangle \otimes \ldots$, with a total of N particles occupying the sites $i, i+1, \ldots, i+N-1$.

What is the expected time taken for a state of the form $a|i, 1\rangle + b|j, 1\rangle$, where $i \neq j$, to collapse into one of its component states? What are the possible resulting states and their probabilities?

What is the expected time taken for a state of the form $a|i, N\rangle + b|i+1, N\rangle$ to collapse into one of its component states? What are the possible resulting states and their probabilities?

What is the expected time taken for a state of the form $a|i, N\rangle + b|i+M, N\rangle$, where $M > N$, to collapse into one of its component states? What are the possible resulting states and their probabilities?

Justify your answers carefully from the standard quantum measurement postulates, explaining any notation you introduce.

Comment briefly on the implications of these results for physical theories in which a spontaneous collapse postulate is added to standard quantum theory.

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