UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 14 June, 2011 1:30 pm to 3:30 pm

PAPER 50

CONTROL OF QUANTUM SYSTEMS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Throughout this exam let $[A, B] = AB - BA$ be the usual matrix commutator and

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

the usual Pauli matrices.

1 Controllability, Reachability, Constructive Control

(a) Briefly define the notions of a bilinear Hamiltonian control system, reachable set and controllability.

(b) Define the notions of pure-state and density operator controllability for quantum systems. Give simple Lie algebraic criteria for both concepts for bilinear Hamiltonian control systems.

(c) Consider the density operators for a system with Hilbert space dimension $N = 2$ (qubit):

$$\rho_0 = \frac{1}{4}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_1 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \rho_2 = \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1}$$

Which of the states are unitarily equivalent?

(d) Assume a two-level system evolves according to the control-dependent dynamical law

$$i\hbar\dot{\rho}(t) = [Z + f(t)X, \rho(t)], \tag{2}$$

where $\rho(t)$ is a density operator defined on $\mathcal{H} = \mathbb{C}^2$. Explain (i) if the systems is controllable and (ii) which of ρ_k defined in (1) are reachable from each other by applying a suitable (open-loop) control function.

(e) Consider a spin network with the Hamiltonian

$$H_0 = \sum_{1 \leqslant m < n \leqslant N} \alpha_{mn}(X_m X_n + Y_m Y_n + \kappa Z_m Z_n) \tag{3}$$

where X_n (respectively, Y_n, Z_n) denotes an N-fold tensor product, all of whose factors are the identity I except for the nth factor, which is the Pauli matrix X (respectively, Y, Z); e.g. for a network with 3 spins we would have $X_2 = I \otimes X \otimes I$. Let $S = \sum_{n=1}^{N} Z_n$.

(i) Using the identities $XY = iZ$, $YZ = iX$, $ZX = iY$, show that $[H_0, S] = 0$.

(ii) Assume we have a control Hamiltonian $H_1 = Z_k$ for some fixed $k \in \{1, \ldots, N\}$. Is the bilinear control system defined by $H[f(t)] = H_0 + f(t)H_1$ controllable? Briefly explain why or why not.

2 Constructive Control, Rotating Frame and Rotating Wave Approximation

(a) Let $U(t)$ be the solution of the Schrodinger equation $i\hbar\frac{d}{dt}U(t) = H[f(t)]U(t)$ with $U(0) = \mathbb{I}$ and $H[f(t)] = H_0 + f(t)H_1$, where H_0, H_1 are Hermitian operators on $\mathcal{H} = \mathbb{C}^N$, and $\mathbb{I}$ is the identity operator on $\mathcal{H}$. Show that the interaction picture evolution operator $U_I(t) = U_0(t)^\dagger U(t)$ with $U_0(t) = \exp(-itH_0/\hbar)$ satisfies

$$i\hbar\frac{d}{dt}U_I(t) = f(t)H_I(t)U_I \qquad \text{with} \qquad H_I(t) = U_0(t)^\dagger H_1 U_0(t). \tag{1}$$

(b) Consider the bilinear Hamiltonian control system given by $H[f(t)] = H_0 + f(t)H_1$, where

$$H_0 = \begin{pmatrix} -\omega_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\omega_2 \end{pmatrix}, \quad H_1 = \begin{pmatrix} 0 & d_1 & 0 \\ d_1 & 0 & d_2 \\ 0 & d_2 & 0 \end{pmatrix}, \quad \omega_1, \omega_2, d_1, d_2 \in \mathbb{R} \tag{2}$$

Choosing units such that $\hbar = 1$, show that the interaction picture Hamiltonian is

$$H_I(t) = \begin{pmatrix} 0 & d_1 e^{-i\omega_1 t} & 0 \\ d_1 e^{i\omega_1 t} & 0 & d_2 e^{i\omega_2 t} \\ 0 & d_2 e^{-i\omega_2 t} & 0 \end{pmatrix}. \tag{3}$$

(c) Assume $f(t) = A_1(t)\cos(\omega_1 t)$ and $\Delta\omega = \omega_2 - \omega_1$. Using (3) and $2\cos(x) = e^{ix} + e^{-ix}$, show that

$$f(t)H_I = \frac{A_1(t)}{2}\left[\begin{pmatrix} 0 & d_1 & 0 \\ d_1 e^{i2\omega_1 t} & 0 & d_2 e^{i(2\omega_1+\Delta\omega)t} \\ 0 & d_2 e^{-i\Delta\omega t} & 0 \end{pmatrix} + \begin{pmatrix} 0 & d_1 e^{-i2\omega_1 t} & 0 \\ d_1 & 0 & d_2 e^{i\Delta\omega t} \\ 0 & d_2 e^{-i(2\omega_1+\Delta\omega)t} & 0 \end{pmatrix}\right]. \tag{4}$$

Explain under what assumptions can we simplify

$$f(t)H_I \approx \frac{A_1(t)d_1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

(d) Assuming the simplifying assumptions in part (c) hold, applying control fields of the form $B_{1,x}(t) = f_1(t)\cos(\omega_1 t)$, $B_{1,y}(t) = f_2(t)\sin(\omega_1 t)$, $B_{2,x}(t) = f_3(t)\cos(\omega_2 t)$, $B_{2,y}(t) = f_4(t)\sin(\omega_2 t)$, respectively, gives rise to a Hamiltonian of the form

$$H = \Omega_1(t)x_{12} + \Omega_2(t)y_{12} + \Omega_3(t)x_{23} + \Omega_4(t)y_{23}, \tag{6}$$

where x_{mn} and y_{mn} are generalized Pauli matrices, i.e., here

$$x_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y_{12} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad y_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}. \tag{7}$$

Explain how we implement any unitary operator in $\mathbb{SU}(3)$ by applying a sequence of simple pulses that effect rotations about x_{12}, y_{12}, x_{23}, y_{23}. Sketch the general algorithm for decomposing a special unitary operator in to a sequence of operations on 2D subspaces and further decompose these rotations using the Euler decomposition.

(e) Explain how we can transfer population from level 1 to level 3 for the three-level system with the effective Hamiltonian (6) above without populating level 2.

3 Optimal & Adaptive Control

(a) Sketch a simple block-diagram of a control system. Explain what is special (compared to the classical case) about the roles the sensors (measurements), actuators and environment play when the system to be controlled is governed by the laws of quantum physics. Briefly explain the difference between open-loop and closed-loop control.

(b) Define suitable objective functions for the problems of quantum state engineering, quantum process (gate) engineering and observable optimization.

(c) Let $\|X\|_{HS} = \sqrt{\mathrm{Tr}(X^\dagger X)}$ be the Hilbert-Schmidt (HS) norm. Show that the HS-distance between two unitary operators U and W on the Hilbert space $\mathcal{H} \simeq \mathbb{C}^N$ satisfies

$$\|W - U\|_{HS}^2 = 2N - 2\mathrm{Re}\,\mathrm{Tr}[W^\dagger U]. \tag{1}$$

(d) Let $H = H_0 + f(t)H_1$ be a bilinear control system, where $f(t)$ is a piecewise constant control function $f(t) = f_k$ for $t_{k-1} < t \leqslant t_k$ with $t_K = T$. Let $U_f(T)$ be the solution of the Schrodinger equation $i\dot{U}(t) = [H_0 + f(t)H_1]U(t)$ for the piecewise constant control $f(t)$ above and $U(0) = \mathbb{I}$. First show that

$$U_f(t, t_{k-1}) := U_f(t_2)U_f(t_1)^\dagger = \exp[-i(t - t_{k-1})(H_0 + f_k H_1)] \tag{2}$$

for $t_k \leqslant t \leqslant t_{k-1}$, then show that

$$\frac{\partial U_f(T)}{\partial f_k} = U_f(t_K, t_{K-1}) \cdots U_f(t_{k+1}, t_k)\, I_k\, U_f(t_{k-1}, t_{k-2}) \cdot U_f(t_1, t_0), \tag{3}$$

where

$$I_k = \int_{t_{k-1}}^{t_k} U_f(t_k, t)(-iH_m)U_f(t, t_{k-1})dt. \tag{4}$$

(e) Use Eq. (3) to derive an expression for the gradient of the normalized error $\mathcal{E}(\mathbf{f}) = \frac{1}{N}\|W - U_f(T)\|_{HS}^2$ with respect to the control variables f_k.

4 Feedback Control & Stabilization

(a) Sketch the basic setup for measurement-based feedback control using homodyne detection.

(b) Consider the reduced Bloch equation for a single qubit subject to direct feedback

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ z(t) \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} (2\lambda+1)^2 & -4\alpha \\ 4\alpha & (2\lambda+1)^2+1 \end{pmatrix}\begin{pmatrix} x(t) \\ z(t) \end{pmatrix} - \begin{pmatrix} 0 \\ 2\lambda+1 \end{pmatrix}. \tag{1}$$

where $x(t) = \mathrm{Tr}(X\rho(t))$, $z(t) = \mathrm{Tr}(Z\rho(t))$ and X, Z are the Pauli operators defined above. Show that except for $(\alpha, \lambda) = (0, -\frac{1}{2})$, the reduced system has a unique steady state given by

$$x_{ss} = -8\alpha(2\lambda+1)/D, \quad z_{ss} = -2(2\lambda+1)^3/D, \tag{2}$$

where $D = (2\lambda+1)^2[(2\lambda+1)^2+1] + 16\alpha^2$.

(c) Show that the state $(x, z) = (\sin\theta_d, \cos\theta_d)$ is a steady state of the system if the driving and feedback strengths α and λ, respectively, are set to

$$\alpha = \frac{1}{4}\sin\theta_d\cos\theta_d, \quad \lambda = -\frac{1}{2}(1+\cos\theta_d). \tag{3}$$

(e) In the standard semi-classical model of quantum control the goal is to control a quantum system using external fields produced by essentially classical actuators and measurements. An alternative approach is to replace the classical controller by another quantum system that acts as a quantum controller. A very simple example of such a system is a cavity that interacts with a quantized external field. Let b_0, b_1 and a be stochastic operators representing the input, output and cavity mode, respectively. It can be shown that for a simple cavity with cavity decay rate γ we obtain the following linear control system

$$\frac{d}{dt}a(t) = -\frac{\gamma}{2}a(t) - \sqrt{\gamma}b_0(t) \tag{4a}$$

$$b_1(t) = \sqrt{\gamma}a(t) + b_0(t). \tag{4b}$$

By taking the Laplace transform of the equations, show that $\tilde{b}_1(s) = G(s)\tilde{b}_0(s)$ with gain function $G(s) = \frac{s-\gamma/2}{s+\gamma/2}$, where $\tilde{b}_0(s) = L[\hat{b}_0(t)](s)$ and $\tilde{b}_1(s) = L[\hat{b}_1(t)](s)$ are the Laplace transforms of $b_0(t)$ and $b_1(t)$, respectively, and apply Nyquist's stability criterion to decide if the cavity-field system is stable.

Hint: The Laplace transform L is linear and satisfies $L[\frac{d}{dt}a(t)](s) = sL[a(t)](s)$ assuming $a(0) = 0$.

END OF PAPER