UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 8 June, 2011 9:00 am to 12:00 pm

PAPER 48

QUANTUM INFORMATION THEORY

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider the following state of four quantum systems

$$|\Psi_p\rangle = \sqrt{p}\,|0000\rangle_{ABCD} + \sqrt{1-p}\,|1111\rangle_{ABCD},$$

where $1/2 < p < 1$.

(i) Assume that Alice, Bob, Clair and Donald share n identical copies of the state $|\Psi_p\rangle$, where n is very large, and want to transform them into $k(n)$ copies of a 4-partite GHZ state $|\Psi_{1/2}\rangle$ using local operations and classical communication only. Sketch an explicit protocol for the asymptotic transformation of $|\Psi_p\rangle$ into $|\Psi_{1/2}\rangle$ and determine the maximal value of the asymptotic rate of conversion, $\lim_{n\to\infty} \dfrac{k(n)}{n}$.

(ii) The parties share n identical copies of $|\Psi_p\rangle$, as in (i), but now they want to transform them into $m(n)$ copies of $|\Psi_q\rangle$, where $1/2 < q < p < 1$. Determine the maximal possible value of the asymptotic rate, $\lim_{n\to\infty} \dfrac{m(n)}{n}$ and briefly explain the idea behind the protocol.

(iii) If the parties shared the following state instead of $|\Psi_p\rangle$

$$|\Phi_p\rangle = \sqrt{\frac{p}{2}}\,|0000\rangle_{ABCD} + \sqrt{\frac{1-p}{2}}\,|0111\rangle_{ABCD} + \sqrt{\frac{p}{2}}\,|1000\rangle_{ABCD} - \sqrt{\frac{1-p}{2}}\,|1111\rangle_{ABCD},$$

would your answers for (i) and (ii) change? Justify.

2

(i) Given 3 random variables A,B, and C using the chain rule for the entropies

$$H(A,B) = H(A) + H(B|A)$$

show that

$$H(B,C|A) = H(C|A) + H(B|C,A).$$

Justify why the following identity also holds

$$H(B,C|A) = H(B|A) + H(C|B,A).$$

(ii) Suppose we make an inference about a random variable X based on knowledge of a random variable Y. From Y we calculate $g(Y) = \widehat{X}$, which is an estimate of X. Suppose X and $\widehat{X}$ both take values in a finite set $\mathcal{X}$. Let E be a random variable defined as follows:

$$E = 1 \quad \text{if } \widehat{X} \neq X, \qquad \text{and} \quad E = 0 \quad \text{if } \widehat{X} = X.$$

Defining $p_e = \text{Prob}(E = 1)$ show that

$$H(X|Y) \leqslant p_e \log(|\mathcal{X}| - 1) + h(p_e),$$

where $|\mathcal{X}|$ denotes the cardinality of the set $\mathcal{X}$ (and is equal to the number of possible values that X takes), and $h(p_e)$ denotes the binary entropy.

[Hints: Use (i) to expand $H(E, X|Y)$, and the fact that conditioning reduces entropy. Note that

$$H(X|E,Y) = \text{Prob}(E = 0)H(X|Y, E = 0) + \text{Prob}(E = 1)H(X|Y, E = 1).]$$

3

(i) Consider the depolarizing channel

$$\Phi(\rho) := (1-p)\rho + \frac{p}{3}\sum_{i=1}^{3} \sigma_k \rho \sigma_k,$$

for $0 \leqslant p \leqslant 3/4$. Here σ_1, σ_2 and σ_3 denote the Pauli matrices σ_x, σ_y and σ_z, respectively:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Prove that under the action of such a channel, the Bloch sphere shrinks uniformly by a factor $\left(1 - \frac{4p}{3}\right)$.

(ii) Show that the depolarizing channel can be equivalently expressed as follows:

$$\Phi(\rho) = \frac{q}{2}I + (1-q)\rho,$$

where I denotes the identity operator and q is a probability. Find the relation between q and p.

(iii) Give the value of p for which

$$(I \otimes \Phi)[|\phi^+\rangle\langle\phi^+|] = \frac{I \otimes I}{4}$$

where $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$.

4

Consider a series of experiments on two separated particles, where Alice and Bob each perform one of two measurements at random. Let us label their measurements as $s = \{0, 1\}$ and $t = \{0, 1\}$ respectively. The measurement outcomes a_s,b_t take values ± 1.

(i) Consider the expectation value $\mathbb{E}$ over many runs of the experiment of the product of their measurement results $\mathbb{E}(a_s b_t)$. If the measurement results can be described by a local hidden variable theory, derive the optimal upper bound on the quantity $\beta = |\mathbb{E}(a_0 b_0) + \mathbb{E}(a_0 b_1) + \mathbb{E}(a_1 b_0) - \mathbb{E}(a_1 b_1)|$.

(ii) Consider the probability distribution $p(a, b|s, t)$ over outcomes a, b given that measurement s, t was performed. What are the conditions on this probability distribution such that signals are not sent between Alice and Bob? I.e. such that Alice's measurement results a do not depend on Bob's measurement settings t and likewise for Bob's results and Alice's settings.

(iii) Write down a probability distribution $p(a, b|s, t)$ that achieves the algebriac maximum $\beta = 4$ but which is still no-signalling.

5

(i) What is the *quantum capacity* of a channel Λ?

(ii) Show that if a pure state $|\psi\rangle_{ABE}$ is close in trace distance to being in a maximally entangled state between Alice and Bob, then Eve must have very little entanglement between herself and the state held by Alice and Bob. Hint: to bound Eve's entropy, you can use Fannes inequality which simply says that if two states are close in trace distance, their entropies must be close: $|S(\rho) - S(\sigma)| \leqslant T \log(d-1) + H[\{T, 1-T\}]$, with $T = \frac{1}{2}||\rho - \sigma||_1$ the trace distance, d is the dimension of the Hilbert space, $S(\cdot)$ is the von-Neumann entropy, and $H(\cdot)$ is the Shannon entropy.

(iii)A symmetric channel Λ_s is a channel which is symmetric with respect to the receiver and the environment of the channel. I.e. the output of the channel ρ_B, given by $\rho_B = \Lambda_s(\psi) = Tr_E(U\psi \otimes |0\rangle\langle 0|_E U^\dagger)$ is equal to $\rho_E = Tr_B(U\psi \otimes |0\rangle\langle 0|_E U^\dagger)$. Show that the quantum capacity of such a channel is zero. Hint: Suppose many copies of the channel can be used to share a maximally entangled state between the sender and receiver. Then use the fact that the channel is symmetric and part (ii).

END OF PAPER