UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 14 June, 2011 9:00 am to 11:00 am

PAPER 47

SOLITONS AND INSTANTONS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Derive the Euler-Lagrange equation of motion associated to the action

$$S_M[\theta] = \frac{1}{2}\int \Big(\theta_t^2 - \theta_x^2 - 2(1-\cos\theta)\Big)\, dxdt\,,$$

where $\theta(t,x) \in \mathbb{R}$ is a real scalar field on two dimensional Minkowski space $\mathbb{R}^2 \ni (t,x)$. Explain the Bogomolny argument, and use it to derive the time independent kink solutions of this equation with boundary conditions $\theta \to 0$ (respectively 2π) as $x \to -\infty$ (respectively $+\infty$).

Prove that S_M is Lorentz invariant and deduce that there are exact solutions representing kinks moving along straight lines $x = ut + x_0$.

Let $A_0 dt + A_1 dx$ be an electromagnetic potential with associated electric field $E = \partial_t A_1 - \partial_x A_0$. Define the gauge transformations and show that E is gauge invariant.

Let the electromagnetic potential be coupled to θ via the action functional

$$S[\theta, A] = S_M[\theta] + \frac{1}{2}\int \Big(E^2 + 2\kappa\epsilon^{\mu\nu}A_\mu\partial_\nu\theta\Big)\, dxdt\,.$$

(Here $\epsilon^{\mu\nu}$ is the antisymmetric symbol with $\epsilon^{01} = 1$ and the summation convention is understood. The coupling constant κ is a fixed number.) Obtain the Euler-Lagrange equations of motion for A, θ. Show that if θ is a gauge invariant field then the equations of motion are gauge invariant.

Is the action functional gauge invariant? Discuss in relation to the gauge invariance of the equations of motion.

2

(i) Define the curvature (or field strength) F associated to an $SU(2)$ connection A on $\mathbb{R}^4$. Write down the action functional for pure $SU(2)$ Yang-Mills theory $S(A)$ on $\mathbb{R}^4$, and define instantons.

(ii) Explain how, under the assumption that there is no dependence on the coordinate x^4, $S(A)$ can be related to the static Yang-Mills-Higgs action functional on $\mathbb{R}^3$:

$$V(a,\Phi) = \int_{\mathbb{R}^3} \Bigg[\langle f_{jk}, f_{jk} \rangle + \langle \nabla_j \Phi + [a_j,\, \Phi], \nabla_j \Phi + [a_j,\, \Phi] \rangle \Bigg] \, d^3x \,.$$

(Here f_{jk} is the field associated to the gauge potential a_j, and Φ is the Higgs field, and all these fields take values in the Lie algebra $su(2)$, on which the inner product $\langle\, \Phi \,,\, \Psi \,\rangle = -\mathrm{tr}\Phi\Psi$ on $su(2)$ is used.)

(iii) Explain the Derrick scaling argument, as it applies to $V(a,\Phi)$ and discuss very briefly the solitons which minimize $V(a,\Phi)$ with condition $|\Phi|^2 \to 1$ as $|x| \to +\infty$, i.e. the BPS monopoles. Do they correspond to instantons under the relation you explained in (ii)?

(iv) Derive the Euler-Lagrange equations of motion from V and hence find a formula for $\Delta|\Phi|^2 = \nabla^2|\Phi|^2$ (when (a,Φ) is a BPS monopole solution). Deduce that $|\Phi|^2 \approx 1 + \frac{c}{|x|}$ for large $|x|$, for some real c. Give a formula for c in terms of the energy $V(a,\Phi)$ of the monopole.

3

Consider the two dimensional static abelian Higgs model on the unit disc $\Sigma = \{x \in \mathbb{R}^2 : |x|^2 < 1\}$, with the hyperbolic metric

$$g = e^{2\rho}((dx^1)^2 + (dx^2)^2)\,, \quad \text{where} \quad e^{2\rho} = 8/((1-|x|^2)^2)\,.$$

The energy is

$$V(A,\Phi) = \frac{1}{2}\int_\Sigma \Big[B^2 + e^{-2\rho}|(\nabla - iA)\Phi|^2 + \frac{1}{4}(1-|\Phi|^2)^2\Big]e^{2\rho}\,d^2x$$

where $\Phi(x) \in \mathbb{C}$ and $A = A_1 dx^1 + A_2 dx^2$ is the magnetic potential with associated magnetic field $B = e^{-2\rho}(\partial_1 A_2 - \partial_2 A_1)$. You may assume that all fields are smooth, and that $|1-|\Phi|^2|$ has limit zero as $|x| \to 1$.

Work out a Bogomolny decomposition for V, and derive a pair of first order equations for A, Φ whose solutions would give minimum energy solutions to the Euler-Lagrange equations.

Show that there is a solution to these equations with $\Phi = f(r)z^N$, where N is an integer and f is a positive radial function, as long as f solves the equation

$$-(\ln f)'' - \frac{1}{r}(\ln f)' = \frac{4}{(1-r^2)^2}(1 - r^{2N}f^2).$$

Give a formula for the corresponding magnetic potential A for this solution.

Show that for $N = 1$ the function $f(r) = \frac{2}{1+r^2}$ is a solution of the equation.

[In this question $z = x^1 + ix^2$ and $r^2 = |z|^2 = |x|^2 = (x^1)^2 + (x^2)^2$.]

4

Consider the equation

$$\partial_t^2 \phi - \Delta\phi + m^2\phi = |\phi|^{p-1}\phi \tag{1}$$

for a complex scalar field $\phi(t,x) \in \mathbb{C}$ on $1+n$ dimensional Minkowski space $\mathbb{R}^{1+n} \ni (t,x)$. The parameters $m > 0$ and $p > 1$ are constant.

Derive an equation for a positive function $f_\omega = f_\omega(x) \in \mathbb{R}$ such that $\phi(t,x) = e^{i\omega t} f_\omega(x)$ is a non-topological soliton solution of (1); assume $\omega^2 < m^2$. Show that

$$f_\omega(x) = (m^2 - \omega^2)^{\frac{1}{p-1}} f(\sqrt{m^2 - \omega^2}\, x) \tag{2}$$

where f satisfies $-\Delta f + f = f^p$; assume that this equation has a smooth solution which decreases exponentially to zero as $|x| \to +\infty$.

Defining $\psi = \phi_t$ write (1) as an equivalent first order in time system of equations for (ϕ, ψ). Write down the energy for this system. Show that the system is invariant under global phase rotation $(\phi, \psi) \mapsto e^{i\chi}(\phi, \psi)$ for any $\chi \in \mathbb{R}$. What is the corresponding Noether conserved quantity Q?

Show that the non-topological solitons can be regarded as stationary points for the energy with the constraint that $Q = q$ is fixed.

State the stability criterion for the non-topological solitons in terms of Q, and obtain a condition on ω (in terms of m, p and n) which implies stability.

END OF PAPER