UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2011 9:00 am to 12:00 pm

PAPER 46

ADVANCED QUANTUM FIELD THEORY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

In the following $\mathcal{L}_0$ denotes the Lagrangian of free, massive scalar field theory,

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi \ -\frac{1}{2}m^2\phi^2$$

where $x_\mu y^\mu = x^\mu y_\mu = \eta_{\mu\nu}x^\mu y^\nu$ with metric convention $\eta = \mathrm{diag}(-1, +1, \ldots, +1)$.

1

(a) A simple harmonic oscillator of angular frequency ω is described by the Hamiltonian

$$\hat{H} \;=\; \frac{\hat{p}^2}{2} \;+\; \frac{1}{2}\omega^2 \hat{q}^2$$

where $\hat{p}$ and $\hat{q}$ are momentum and position operators for a particle moving in one dimension which obey the canonical commutation relations $[\hat{p}, \hat{q}] = -i\hbar$.

Consider the two-point function

$$H^{(2)}(t_1, t_2) \;=\; \langle 0 \,|\mathrm{T}\,\{\hat{q}(t_1)\;\hat{q}(t_2)\}|\,0\rangle$$

where $|0\rangle$ is the groundstate of the system, $\hat{q}(t)$ is the position operator in the Heisenberg picture and T denotes time-ordering. You may assume that $H^{(2)}$ is given by the path integral formula

$$H^{(2)}(t_1, t_2) \;=\; \frac{\int [dq(t)]\;\, q(t_1)\, q(t_2)\; \exp\left(\frac{i}{\hbar} S[q]\right)}{\int [dq(t)]\;\, \exp\left(\frac{i}{\hbar} S[q]\right)}\,,$$

where $S[q]$ is the corresponding classical action of a path $q(t)$ in the time interval $-\infty < t < +\infty$.

By defining a suitable generating functional, evaluate $H^{(2)}(t_1, t_2)$, giving the answer in the form of an integral over the energy of the particle, choosing the contour so that the path integral formula above is convergent. Thus show that

$$\begin{aligned} H^{(2)}(t_1, t_2) \;&=\; -\frac{\hbar}{2\omega} \exp\left(-i\omega\,(t_1 - t_2)\right) \qquad t_1 > t_2 \\ &=\; +\frac{\hbar}{2\omega} \exp\left(+i\omega\,(t_1 - t_2)\right) \qquad t_1 < t_2\,. \end{aligned}$$

[10 marks]

(b) The effective action $\Gamma[\varphi]$ is defined in terms of the generating functional $W[J]$ for connected diagrams as

$$\Gamma[\varphi] \;=\; \int d^d x\, \varphi(x)\, J(x) \;\; - \;\; W[J]\,.$$

Show that

$$\int d^d y \;\frac{\delta^2 \Gamma[\varphi]}{\delta\varphi(x)\delta\varphi(y)}\;\frac{\delta^2 W[J]}{\delta J(y)\delta J(z)} \;=\; \delta^{(d)}(x-z)\,.$$

Consider the quantity

$$\Gamma_3(x_1,x_2,x_3) \;=\; -i\,\frac{\delta^3\Gamma[\varphi]}{\delta\varphi(x_1)\delta\varphi(x_2)\delta\varphi(x_3)}\bigg|_{\varphi\equiv 0}$$

By deriving an equation relating Γ_3 to two- and three-point connected Green's functions, and illustrating the equation graphically, explain why it gets contributions only from *one-particle irreducible* diagrams with three (amputated) external legs. [10 marks]

2

A scalar field theory in d spacetime dimensions has Lagrangian

$$\mathcal{L} = \mathcal{L}_0 \; - \; \frac{\lambda_{2n}}{(2n)!}\,\phi^{2n}$$

where n is a positive integer. Write down (without derivation) a set of Feynman rules for calculating the k-point Euclidean momentum space Green's function $F_k(p_1, \ldots, p_k)$. [4 marks]

The amputated k-point function $\hat{F}_k$ is defined by the equation

$$F_k\,(p_1, p_2, \ldots, p_k) = \\ i\,(2\pi)^d\,\delta^{(d)}\left(\Sigma_{j=1}^{k}\; p_j\right) \times \Pi_{j=1}^{k}\left(\frac{i}{p_j^2+m^2}\right) \times \hat{F}_k\,(p_1, p_2, \ldots, p_k)\;.$$

(a) For ϕ^4 theory (i.e., $n = 2$) in $d = 4$, evaluate the one-loop contribution to $\hat{F}_4(p_1, p_2, p_3, p_4)$ with a momentum space cut-off Λ giving the answer as an integral over an undetermined loop momentum. (You can ignore any one-particle reducible and/or disconnected contributions). Find the divergent part of the resulting expression. [6 marks]

(b) For an arbitrary positive integer n, draw the connected Feynman diagram which contributes to the two-point function $\hat{F}_2(p_1, p_2)$ at linear order in λ_{2n} and evaluate it in Dimensional Regularization. Find the *leading* divergence in four-dimensions as a pole in $\epsilon = 4 - d$. [10 marks]

[In this question you may use the following formula for the volume of a unit $d-1$ sphere:

$$\text{Vol}\left(S^{d-1}\right) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$$

where the Euler Gamma-function is defined as

$$\Gamma(\alpha) = \int_0^\infty dx\, x^{\alpha-1}\, e^{-x}$$

for $\text{Re}[\alpha] > 0$ and obeys $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ for all complex values of α.]

3

(a) Define the superficial degree of divergence, D, of a Feynman graph in a scalar field theory in d spacetime dimensions. Which of the following two statements is true?

(i) If $D < 0$ then the diagram is finite.

(ii) If $D \geqslant 0$ then the diagram is divergent.

Justify your answer by analysing the divergences of a generic Feynman diagram.

Consider a Feynman diagram with L loops, E external legs and V_n vertices of valence n in a scalar field theory in d spacetime dimensions. Derive a formula for D in terms of L, E, V_n, n and d only. [8 marks]

(b) In the following you may assume any part of the *Renormalisation Theorem* provided it is clearly stated. Consider a theory in d spacetime dimensions with Lagrangian

$$\mathcal{L} = \mathcal{L}_0 - V(\phi) .$$

In the following cases identify the relevant primitively divergent Feynman diagrams and state the counterterm(s) required to remove the divergence. [*N.B. You are not required to find the coefficient of the counterterm only its general form.*]

(i) $V(\phi) = \lambda_6 \phi^6/6!$ in $d = 4$. Include diagrams of order λ_6 only.

(ii) $V(\phi) = \lambda_4 \phi^4/4!$ in $d = 6$. Include one-loop diagrams only. [8 marks]

(c) The β-function of $SU(N)$ Yang–Mills theory is given as

$$\beta(g) \;=\; \mu\frac{dg(\mu)}{d\mu} \;=\; -\frac{11}{3}\,N\,\frac{g^3}{(4\pi)^2}\,.$$

Determine the running coupling $g(\mu)$ in terms of its value $g(\mu_0)$ at some reference scale $\mu = \mu_0$.

Define a dynamical scale Λ_{YM} such that $g(\mu) = 1$ for $\mu = \Lambda_{YM}$ and show that it is independent of μ_0. [4 marks]

4

Write an essay on non-Abelian gauge theory. Your account should include a discussion of the classical Yang–Mills Lagrangian and its gauge invariance. You should give the gauge-fixed Lagrangian for gauge group $SU(N)$ in the family of covariant linear gauges $\partial_\mu A^\mu = f(x)$, where f is an undetermined Lie algebra valued function, carefully discussing the origin of each term in the Lagrangian. Finally you should discuss the schematic form of the resulting Feynman rules. [20 marks]

END OF PAPER