UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 9 June, 2011 1:30 pm to 4:30 pm

PAPER 45

THE STANDARD MODEL

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A field theory for real classical scalar fields ϕ_i, $i = 1, \ldots n$, has a Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu \phi_i \partial_\mu \phi_i - \tfrac{1}{8} g(\phi_i\phi_i - v^2)^2 \,,$$

which is invariant under the symmetry group $SO(n)$. Explain how this symmetry is spontaneously broken. What is the unbroken symmetry? What are the masses of the particles in the quantum field theory, neglecting any quantum corrections?

Let $n = 3$ and consider the associated gauge theory involving a gauge field $A_{\mu i}$ so that

$$\partial_\mu \phi_i \to D_\mu \phi_i = \partial_\mu \phi_i + e\, \epsilon_{ijk} A_{\mu j} \phi_k \,.$$

Write down an expression for the gauge covariant field strength $F_{\mu\nu\, i}$ and explain why $\mathcal{L}_{\text{gauge}} = -\frac{1}{4} F^{\mu\nu}{}_i F_{\mu\nu\, i}$ is gauge invariant.

Choosing in the ground state $\phi_3 = v$, $\phi_1 = \phi_2 = 0$, show that the theory can be re-expressed in terms of a massive complex vector field W_μ and a massless gauge field A_μ. If A_μ is regarded as corresponding to the photon what are the electric charges of the massive vectors?

This spontaneously broken gauge theory was once considered as part of an alternative to the standard model. Explain briefly why the results obtained might make this appropriate for forming a combined theory of weak and electromagnetic interactions. In what crucial respects does this theory differ from the corresponding part of the standard model?

2

In the standard model the coupling of leptons to the charged W-field is given by

$$\mathcal{L}_I = \frac{g}{2\sqrt{2}}(J^\mu W_\mu + \text{hermitian conjugate})\,,$$
$$J^\mu = \overline{\nu}_e\gamma^\mu(1-\gamma_5)e + \overline{\nu}_\mu\gamma^\mu(1-\gamma_5)\mu + \overline{\nu}_\tau\gamma^\mu(1-\gamma_5)\tau\,.$$

Describe briefly how this leads at low energies to the standard form for lepton weak interactions with an overall coupling $G_F/\sqrt{2} = g^2/8m_W^2$.

Assuming

$$\langle 0|W_\mu(0)|W^+(p)\rangle = \epsilon_\mu(p)\,, \qquad \sum_{W\ \text{spins}} \epsilon_\mu(p)\epsilon_\nu(p)^* = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m_W^2}\,,$$

calculate the total decay rate for $W^+ \to e^+\nu_e$, neglecting m_e,

$$\Gamma_{W^+\to e^+\nu_e} = \frac{G_F}{\sqrt{2}}\,\frac{m_W^3}{6\pi}\,.$$

Explain briefly why $\Gamma_{W^+\to e^+\nu_e} \approx \Gamma_{W^+\to\mu^+\nu_\mu}$ although $\Gamma_{K^+\to e^+\nu_e} \ll \Gamma_{K^+\to\mu^+\nu_\mu}$.

Assuming $\Gamma_{W^+\to\text{hadrons}} = \Gamma_{W^+\to\text{quarks}}$ if $m_q \ll m_W$ what is the expected value for $\Gamma_{W^+\to\text{hadrons}}/\Gamma_{W^+\to\text{leptons}}$, disregarding any mixing angles?

[The formula for the decay width of a particle with mass m is

$$\Gamma = \frac{1}{2m}\sum_X (2\pi)^4\delta^4(p-p_X)\,|\langle X|\mathcal{L}_I|p\rangle|^2\,, \quad \sum_X = \prod_{\text{momenta}} \int \frac{\mathrm{d}^3p}{(2\pi)^3 2p^0} \sum_{\text{spins}}\,.$$

You may also use $\mathrm{tr}(\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta) = 4(g_{\alpha\beta}\,g_{\gamma\delta} + g_{\alpha\delta}\,g_{\beta\gamma} - g_{\alpha\gamma}\,g_{\beta\delta})$.]

3

For the standard model with the gauge group $SU(2)_T \times U(1)_Y$ describe the experimental reasons why only left handed fermion fields belong to non trivial representations of $SU(2)_T$. Acting on a field ψ the generators of $SU(2)_T \times U(1)_Y$ are $\mathbf{T}, Y$. Explain briefly why the gauge covariant derivative acting on ψ has the form

$$D_\mu \psi = (\partial_\mu - ig\,\mathbf{T}\cdot\mathbf{A}_\mu - ig'\,YB_\mu)\psi\,,$$

with g, g' independent couplings.

Let

$$q_+ = \begin{pmatrix} q_{+1} \\ \vdots \\ q_{+n} \end{pmatrix}, \qquad q_- = \begin{pmatrix} q_{-1} \\ \vdots \\ q_{-n} \end{pmatrix},$$

be two sets of n fermion fields such that $\frac{1}{2}(1-\gamma_5)\begin{pmatrix} q_+ \\ q_- \end{pmatrix}$ belongs to a $T=\frac{1}{2}$, $Y=y$ representation while $q_{\pm R} = \frac{1}{2}(1+\gamma_5)q_\pm$, form $T=0$, $Y = y \pm \frac{1}{2}$, representations. Write down the general gauge invariant coupling which is quadratic in the fermion fields to a two-component complex scalar field $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$, which forms a $T=\frac{1}{2}$, $Y=\frac{1}{2}$ representation.

[You may assume that $\phi' = i\tau_2\phi^*$ belongs to a $T=\frac{1}{2}, Y=-\frac{1}{2}$ representation.]

Assume that there is spontaneous symmetry breaking so that in the ground state $\langle\phi\rangle$ is non zero. Explain why we may choose $\langle\phi_1\rangle = 0$, $\langle\phi_2\rangle = v = v^*$. Verify that then $SU(2)_T \times U(1)_Y \to U(1)_Q$ where $Q = T_3 + Y$

Show how coupling of the fermion fields to ϕ generates mass terms. What is the appropriate value for y for quarks? Show how the physical quark masses are obtained and that the coupling of the quarks to the massive charged W's then involves a unitary matrix V. Explain why for $n=2$, V can be taken to depend on a single angle θ_C whereas $n=3$ is necessary to have CP violation through the effects of this interaction?

4

In a massless field theory, a finite amplitude F for a physical process involving an energy E is obtained in perturbation theory by taking the cut off M to infinity in the Feynman integral expression for the unrenormalised amplitude F_0 according to

$$F_0(g_0, M; E) \sim ZF(g, E/\mu) \quad \text{as} \quad M \to \infty\,,$$

for suitable $Z(M/\mu, g)$ and $g_0(M/\mu, g)$, where g_0 is the initial unrenormalised coupling, g is a finite physical coupling and μ is an arbitrary renormalisation scale. Define the β function and obtain the equation

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \gamma(g)\Big)F(g, E/\mu) = 0 \;\; \text{for} \;\; \mu\frac{\mathrm{d}}{\mathrm{d}\mu}Z\bigg|_{g_0} = Z\gamma(g)\,.$$

Show that this has a solution

$$F(g, E/\mu) = \exp\Big(- \int_{g(E)}^{g} \mathrm{d}x\, \frac{\gamma(x)}{\beta(x)}\Big) F(g(E), 1)\,, \qquad \int_{g(E)}^{g} \mathrm{d}x\, \frac{1}{\beta(x)} = -\ln\frac{E}{\mu}\,.$$

What does it mean to say that the theory has a UV fixed point?

Suppose that $\beta(g) = -bg^3$, $b > 0$. Show that $1/g(E)^2 = 2b\ln\frac{E}{\Lambda}$ for some scale Λ. What is the UV fixed point in this case?

Suppose that perturbative calculations give, as well as $\beta(g) = -bg^3$,

$$F(g, E/\mu) = 1 + g^2\Big(C\ln\frac{E}{\mu} + D\Big) + \mathrm{O}(g^4)\,,$$

where C, D are constants. Show that we must have $\gamma(g) = Cg^2 + \mathrm{O}(g^4)$. Obtain the leading behaviour of $F(g, E/\mu)$ as $E \to \infty$ both for $C \neq 0$ and $C = 0$.

Describe briefly how this is relevant in QCD and also in the possibility of embedding the standard model, with gauge group $SU(3)_{\text{colour}} \times SU(2)_T \times U(1)_Y$, in a larger gauge group with a single coupling.

END OF PAPER