UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 6 June, 2011 1:30 pm to 4:30 pm

PAPER 43

SYMMETRIES AND PARTICLES

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

You may use the property of Pauli matrices, $\sigma_i\,\sigma_j \;=\; \delta_{ij}\,I + i\,\epsilon_{ijk}\,\sigma_k$.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The generators of the isospin algebra satisfy

$$[I_3, I_\pm] = \pm I_\pm \,, \quad [I_+, I_-] = 2I_3 \,.$$

What is meant by a highest weight state, $|jj\rangle$, in a representation of this algebra?

Determine the finite set of normalized states (i.e., states of unit norm), $|jm\rangle$, in the isospin-j representation obtained by acting on $|jj\rangle$ a number of times with I_-.

Explain how the tensor operator, T_{kq}, transforms under isospin, where k is the total isospin and $q = -k, -k+1, \ldots, k$.

State the Wigner–Eckart theorem for the matrix element

$$\langle \alpha', j'm' | T_{kq} | \alpha, jm \rangle$$

of T_{kq} between states of total isospin j and j' with eigenvalues of I_3 equal to m and m', respectively. The symbols α and α' denote other quantum numbers.

Explain why this element is non-zero only if $j' \in \{j{+}k, j{+}k{-}1, \ldots, |j{-}k|{+}1, |j{-}k|\}$.

The Δ states are baryons of total isospin $3/2$. Explain why, according to their description in terms of quarks, they have total angular momentum $3/2$ (assuming the quarks in the Δ do not carry orbital angular momentum).

Making use of the Wigner–Eckart theorem find the ratios of the decay widths

$$\frac{\Gamma(\Delta^+ \to n\pi^+)}{\Gamma(\Delta^+ \to p\pi^0)} \,, \qquad \frac{\Gamma(\Delta^0 \to n\pi^0)}{\Gamma(\Delta^0 \to p\pi^-)} \,,$$

where $\pi^\pm$, π^0 are pions and $p \,, n$ are the proton and neutron.

[The decay width for a process $I \to J$, $\Gamma(I \to J)$, is proportional to the square of the amplitude, $|A(I \to J)|^2$.]

2

Briefly explain what is meant by a *Lie algebra*.

Consider the matrix group $GL(n)$ consisting of $n \times n$ matrices, M, that have an expansion close to the identity of the form $M = I + X + O(X^2)$. The complex matrix X is in the Lie algebra $\mathcal{L}_n$ and can be written as $X = (R^i{}_j)\, X^j{}_i$ and the n^2 basis matrices $\{R^i{}_j : 1 \leqslant i, j \leqslant n\}$ are defined by

$$(R^i{}_j)^p{}_q = \delta^{ip}\delta_{qj}\,,$$

$(1 \leqslant p, q \leqslant n)$ so the matrix $R^i{}_j$ has 1 in the i'th row and j'th column and is zero elsewhwere.

Determine the commutation relations of the $R^i{}_j$'s and hence determine the structure constants in this basis.

By considering $[X, R^i{}_j]$ and using the general definition of the adjoint representation show that the adjoint representation of X has the form

$$(X^{ad})^l{}_k{}^i{}_j = X^i{}_k\, \delta^l{}_j - X^l{}_j\, \delta^i{}_k\,.$$

Show that the Killing form $K(X,Y) = \mathrm{tr}(X^{ad}Y^{ad})$ can be written as

$$K(X,Y) = 2\left(n \sum_{i,j} X^j{}_i\, Y^i_j - \sum_i X^i{}_i \sum_j Y^j_j \right)\,.$$

What conditions need to be imposed on the matrices $X^i{}_j$ in order for the group corresponding to $\mathcal{L}_n$ to be:

(a) $U(n)$; (b) $SU(n)$; (c) $SO(n)$?

How can you see from the expression for the Killing form that the group of $n \times n$ matrices, $U(n)$, contains an invariant abelian subgroup, whereas $SU(n)$ does not?

Show that upper triangular matrices of the form

$$X(x,y,z) = \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}\,,$$

with real x, y and z, form a subalgebra of $\mathcal{L}_3$.

What is the element of $GL(3)$ obtained by exponentiating $X(x,y,z)$?

3

Group multiplication for a Lie Matrix group, G, may be expressed as the relation $g(\mathbf{a})\, g(\mathbf{b}) = g(\mathbf{c})$, so that $\mathbf{c}$ defines a point on the group manifold that is reached by multiplying the element at $\mathbf{a}$ by that at $\mathbf{b}$. Show that the infinitesimal shift $\mathbf{a} \to \mathbf{c} = \mathbf{a} + \mathrm{d}\mathbf{a}$ produced by an infinitesimal transformation $\mathrm{d}\mathbf{b}$ close to the origin has the form

$$\mathrm{d}a_i = \mathrm{d}b_j \mu^j{}_i(\mathbf{a}) \,,$$

for $i, j = 1, \ldots, n$ where n is the dimension of G.

What is meant by a G-invariant function, $f(g)$?

Show that for a n-dimensional Lie group G the G-invariant integral of a G-invariant function, f, over the group manifold has the form

$$\int_G d\rho(\mathbf{a})\, f(g(\mathbf{a})) \,,$$

where $g(\mathbf{a}) \in G$ and the form of the measure $\mathrm{d}\rho(\mathbf{a})$ should be determined.

The group $GL(2, \mathbb{R})$ consists of real 2×2 matrices that can be expressed as

$$A = a_0 + a_1\, \sigma_1 + a_2\, \sigma_2 + a_3\, \sigma_3 \,,$$

where σ_i $(i = 1, 2, 3)$ are the Pauli matrices and coefficients a_0, a_1 are real. Find the condition satisfied by these coefficients for the matrices to define the group $SL(2, \mathbb{R})$ of real 2×2 matrices with unit determinant.

Determine the matrix $\mu^j{}_i(\mathbf{a})$ for this group.

Show that the invariant integration measure for the group $SL(2, \mathbb{R})$ has the form

$$\mathrm{d}\rho(\mathbf{a}) = \frac{1}{|a_0|}\, \mathrm{d}^3 a_i \,.$$

Hence, show that the group $SL(2, \mathbb{R})$ is non-compact.

4

The Lie algebra $\mathcal{L}$ of a semi-simple group G has a maximally commuting subalgebra generated by $\underline{H} = (H_1, \ldots, H_r)$. The remaining generators $E_{\underline{\alpha}}$, $E_{\underline{\beta}}$, ..., correspond to roots $\underline{\alpha}$, $\underline{\beta}$, ..., where, for any root $\underline{\alpha}$, $[H_i, E_{\underline{\alpha}}] = \alpha_i \, E_{\underline{\alpha}}$. Furthermore,

$$[E_{\underline{\alpha}}, E_{\underline{\beta}}] = c_{\underline{\alpha}\,\underline{\beta}} \, E_{\underline{\alpha}+\underline{\beta}} \quad (\underline{\alpha} + \underline{\beta} \neq 0)\,,$$

where $c_{\underline{\alpha}\,\underline{\beta}} \neq 0$ if $\underline{\alpha} + \underline{\beta}$ is a root and $c_{\underline{\alpha}\,\underline{\beta}} = 0$ otherwise.

Prove that if $\underline{\alpha}$ is a root then so is $-\underline{\alpha}$.

Show how, with a suitable choice of normalisation, the algebra can be expressed in the form

$$[H_{\underline{\alpha}}, H_{\underline{\beta}}] = 0\,, \qquad [E_{\underline{\alpha}}, E_{-\underline{\alpha}}] = H_{\underline{\alpha}}\,, \qquad [H_{\underline{\alpha}}\,, E_{\underline{\beta}}] = \frac{2\underline{\alpha}\cdot\underline{\beta}}{|\underline{\alpha}|^2} \, E_{\underline{\beta}}\,,$$

where

$$H_{\underline{\alpha}} = \frac{2\underline{\alpha}\cdot\underline{H}}{\underline{\alpha}^2}\,,$$

and the scalar product is with respect to the Killing form.

Suppose that $[E_{\underline{\alpha}}\,, E_{\underline{\beta}+n_+\underline{\alpha}}] = 0$ for some value of n_+, where $\underline{\alpha}$ and $\underline{\beta}$ are roots. Explain why the finite set of roots (i.e., the root string) $\{E_{\underline{\beta}-n_-\underline{\alpha}},\, E_{\underline{\beta}-(n_--1)\underline{\alpha}}, \ldots, E_{\underline{\beta}+n_+\underline{\alpha}}\}$ furnishes a finite-dimensional representation of $SU(2)$, where

$$n_- = n_+ + \frac{2\underline{\alpha}\cdot\underline{\beta}}{\underline{\alpha}^2}\,.$$

Show that the angle θ between two root vectors, $\underline{\alpha}$ and $\underline{\beta}$, is constrained so that

$$0 \leqslant mn = 4\cos^2\theta \leqslant 4\,, \qquad \text{and} \qquad \frac{|\underline{\alpha}|^2}{|\underline{\beta}|^2} = \frac{|m|}{|n|}\,,$$

where m, n are integers.

Define the *simple* roots, $\underline{\alpha}_{(i)}$ and show that $\underline{\alpha}_{(1)} \cdot \underline{\alpha}_{(2)} < 0$, where $\underline{\alpha}_{(1)}$ and $\underline{\alpha}_{(2)}$ are two distinct simple roots.

The rank two Lie algebra of the 14-dimensional simple group G_2 has simple roots $\underline{\alpha}_{(1)} = (1, 0)$ and $\underline{\alpha}_{(2)} = (-3, \sqrt{3})/2$ (in a Cartesian basis). Use this information to sketch the root diagram for G_2.

END OF PAPER