UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2011 9:00 am to 12:00 pm

PAPER 42

QUANTUM FIELD THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State and prove Noether's theorem in the context of a classical Lagrangian field theory defined in Minkowski space.

Yukawa theory has a Lagrangian density

$$\mathcal{L} \;=\; \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi \;-\; \frac{1}{2}m^2\phi^2 \;+\; \overline{\psi}(i\gamma^\mu\partial_\mu - M)\psi \;-\; g\overline{\psi}\psi\phi$$

where ϕ is a scalar field and ψ is a Dirac field. Find the Noether current and conserved charge associated with the phase rotation $\psi \to e^{-i\alpha}\psi$. Find the Noether current and conserved charge associated with time translation symmetry.

Briefly describe any additional Noether currents and charges that this theory has.

2

Free scalar field theory in momentum space is defined by a set of Schrödinger picture operators $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$ satisfying

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}] = [a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}] = 0\,,$$

$$[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}] = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\,,$$

and a Hamiltonian

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}\,,$$

where $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$.

Define what is meant by a Heisenberg picture operator, and find the Heisenberg picture operators $a^H_{\mathbf{p}}$ and $a^{H\dagger}_{\mathbf{p}}$ obtained from $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$. Given the Schrödinger picture fields

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{i\mathbf{p}.\mathbf{x}} + a^\dagger_{\mathbf{p}} e^{-i\mathbf{p}.\mathbf{x}}\right),$$

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} e^{i\mathbf{p}.\mathbf{x}} - a^\dagger_{\mathbf{p}} e^{-i\mathbf{p}.\mathbf{x}}\right),$$

find the corresponding Heisenberg picture fields $\phi^H(x)$ and $\pi^H(x)$, where x denotes $(\mathbf{x}, t)$. Show that ϕ^H obeys the Klein-Gordon equation.

Define the vacuum state $|0\rangle$ and find an expression for $\langle 0|\phi^H(x)\phi^H(y)|0\rangle$ as a 3-momentum integral.

3

The matrices γ^μ satisfy the Dirac algebra

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}1$$

and the hermiticity property $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$. Let u and v be arbitrary Dirac spinors. Show that $(\bar{u}\gamma^\mu v)^* = \bar{v}\gamma^\mu u$. Using the Dirac algebra, determine

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) \quad \text{and} \quad \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\tau)\,.$$

Now let $\{u_s(p)\,,\ s = \pm\frac{1}{2}\}$ and $\{v_s(p)\,,\ s = \pm\frac{1}{2}\}$ be an orthonormalized basis set of spinors satisfying

$$(\gamma\cdot p - m)u_s(p) = 0$$

$$(\gamma\cdot p + m)v_s(p) = 0$$

and

$$\overline{u}_s(p)u_{s'}(p) = 2m\,\delta_{ss'}\,,\ \overline{v}_s(p)v_{s'}(p) = -2m\,\delta_{ss'}\,,\ \overline{u}_s(p)v_{s'}(p) = 0\,.$$

Why is it necessary that $p^2 = m^2$ here? Establish the spin sums

$$\sum_s u_s(p)\overline{u}_s(p) = \gamma\cdot p + m$$

$$\sum_s v_s(p)\overline{v}_s(p) = \gamma\cdot p - m\,.$$

Use the results above to simplify the expression

$$\sum_{s,r}(\overline{u}_s(p)\gamma^\mu v_r(q))^*\,(\overline{u}_s(p)\gamma^\nu v_r(q))\,,$$

where $p^2 = q^2 = m^2$. Describe briefly how this quantity could arise in a QED calculation.

4

Write down the Feynman rules for a QED scattering amplitude involving **only photons** on external lines.

a) Explain the constraints on the photon polarization vectors and how they arise.

b) Explain the key steps in the derivation of the photon propagator.

c) Draw the lowest order Feynman diagram for two photon to two photon scattering, and give the integral expression for the scattering amplitude to this order. (Simplification of the expression is **not** required.)

d) Draw the diagrams that contribute at next to lowest order to this scattering amplitude.

END OF PAPER