UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2011 1:30 pm to 3:30 pm

PAPER 41

STATISTICAL FIELD THEORY

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Explain briefly what is meant by a phase diagram. Give an example of a three dimensional phase digram which contains a tricritical point and describe the nature of the different transitions which may occur.

Give an account of the Landau-Ginsberg (LG) theory of phase transitions in the context of a scalar field theory which should include a discussion of the following points:

(i) the idea of an order parameter;

(ii) the distinction between first-order and continuous phase transitions and how their defining properties are explained;

(iii) the reason why a line of first order transitions must terminate in a critical point associated with a continuous phase transition;

(iv) the idea of *critical exponents* and how they may be derived;

(v) the explanation of the features of the three-dimensional diagram containing a tricritical point.

Explain briefly the idea of the mean field approach and why it is only likely to be valid in the limit of large dimension D.

The Ising model in D dimensions is defined on a cubic lattice Λ with N sites and with spin $\sigma_{\mathbf{n}}$ on the site at position $\mathbf{n}$. The Hamiltonian with zero external magnetic field is

$$\mathcal{H}(\{\sigma\}) = -J \sum_{\mathbf{n}\in\Lambda,\mu} \sigma_{\mathbf{n}}\sigma_{\mathbf{n}+\mu},$$

where $J > 0$ and the sum on μ is over the basis vectors of the lattice Λ. Using the mean-field approximation find an expression for the free energy $A(M)$ of this model, where M is the magnetization which should be defined. Hence show that, in this case, mean field theory predicts a second order phase transition at temperature $T_C = 2DJ/k_B$ where k_B is Boltzmann's constant.

2

A spin model in D dimensions is defined on a cubic lattice of spacing a with N sites and with spin $\sigma_{\mathbf{n}}$ on the $\mathbf{n}$-th site. The Hamiltonian is defined in terms of a set of operators $O_i(\{\sigma\})$ by

$$\mathcal{H}(\boldsymbol{u},\sigma) \;=\; \sum_i u_i O_i(\{\sigma\}) \;,$$

where the u_i are coupling constants with $\boldsymbol{u} = (u_1, u_2, \ldots)$. In particular, $\mathcal{H}$ contains the term $-h\sum_{\mathbf{n}} \sigma_{\mathbf{n}}$ where h is the magnetic field. The partition function is given by

$$\mathcal{Z}(\boldsymbol{u}, C, N) \;=\; \sum_\sigma \exp(-\beta\mathcal{H}(\boldsymbol{u},\sigma) - \beta NC) \;.$$

Define the two-point correlation function $G(\boldsymbol{r})$ for the theory and state how the correlation length ξ parametrizes its behaviour as $|\boldsymbol{r}| \to \infty$. State how the susceptibility χ can be expressed in terms of $G(\boldsymbol{r})$.

Explain how the renormalization group (RG) transformation may be defined in terms of a blocking kernel which, after p iterations, yields a blocked partition function $\mathcal{Z}(\boldsymbol{u}_p, C_p, N_p)$ which predicts the same large-scale properties for the system as does $\mathcal{Z}(\boldsymbol{u}, C, N)$. State how a and N rescale in terms of the RG scale factor b.

Derive the RG equation for the free energy $F(\boldsymbol{u}_p, C_p)$, and explain how it may be expressed in terms of a singular part, $f(\boldsymbol{u})$, which obeys the RG equation

$$f(\boldsymbol{u}_0) \;=\; b^{-pD}\, f(\boldsymbol{u}_p) \;+\; \sum_{j=0}^{p-1} b^{-jD}\, g(\boldsymbol{u}_j) \;, \qquad p > 0 \;.$$

What is the origin of the function $g(\boldsymbol{u})$ which determines the inhomogeneous part of the transformation?

Explain the idea of a fixed point, *relevant* and *irrelevant* operators, a critical surface and a repulsive trajectory in the context of the RG equations. Sketch some typical RG flows near to a critical surface.

Show how the critical exponents characterizing a continuous phase transition may be derived. In the case where there are two relevant couplings $t = (T - T_C)/T_C$ and h, derive the scaling hypothesis for the singular part, F_s, of the free energy:

$$F_s \;=\; |t|^{D/\lambda_t} f_\pm\left(\frac{h}{|t|^{\lambda_h/\lambda_t}}\right) \;,$$

where the meanings of λ_t, λ_h should be explained.

The following critical exponents ν, γ, α are defined for $h = 0$:

$$\xi \sim |t|^{-\nu}, \quad \chi \sim |t|^{-\gamma}, \quad C_V \sim |t|^{-\alpha},$$

where χ is the susceptibility and C_V is the specific heat at constant volume. Establish the scaling relation $\alpha = 2 - D\nu$.

According to the scaling hypothesis the correlation function for $|\boldsymbol{r}| \ll \xi$ takes the form

$$G(|\boldsymbol{r}|) = \frac{1}{|\boldsymbol{r}|^{D-2+\eta}} f_G(|\boldsymbol{r}|/\xi).$$

What form is $G(\boldsymbol{r})$ expected to take when $|\boldsymbol{r}| \gg \xi$? From this parametrization obtain an expression for the susceptibility and derive the scaling relation $\gamma = (2 - \eta)\nu$.

3

A statistical system in D dimensions at temperature T is described by a scalar field theory whose effective Hamiltonian is defined by

$$\begin{aligned} H(\Lambda,\phi) &= \int_{\Lambda^{-1}} d^D x\, \mathcal{H}(\Lambda,\phi(\boldsymbol{x}))\ , \\ \mathcal{H}(\Lambda,\phi) &= \frac{1}{2}(\nabla\phi(\boldsymbol{x}))^2 + \frac{1}{2}m^2(\Lambda,T)\phi^2(\boldsymbol{x}) + \frac{1}{4!}g(\Lambda,T)\phi^4(\boldsymbol{x}) + \dots\ , \end{aligned}$$

where Λ is the ultra-violet cut-off and $\mathcal{H}$ is the Hamiltonian density. The magnetic field is set to zero. The partition function is

$$\mathcal{Z} \;=\; \int \{d\phi\} e^{-H(\Lambda,\phi)}\ .$$

Why do the coupling constants depend on Λ? Why is it reasonable to associate $m(0,T)$ with the correlation length?

In the case of a ϕ^4 scalar field theory, the connected two-point function $G(\boldsymbol{x})$ and its Fourier transform $\tilde{G}(\boldsymbol{p})$ are defined by

$$G(\boldsymbol{x}) \;=\; \langle \phi(0)\phi(\boldsymbol{x})\rangle_c\ , \qquad \tilde{G}(\boldsymbol{p}) \;=\; \int \frac{d^D p}{(2\pi)^D}\, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} G(\boldsymbol{x})\ .$$

Explain what is meant by the term ‘connected’ in this context.

Starting from the definition of the partition function explain briefy how the perturbation expansion to 1-loop order for $G(\boldsymbol{x})$ is established as an expansion in g, the ϕ^4 coupling constant.

State what is meant by the truncated two-point function $\tilde{\Gamma}(\boldsymbol{p})$, and explain how in perturbation theory $\tilde{\Gamma}(\boldsymbol{p})$ may be written as

$$\tilde{\Gamma}(\boldsymbol{p}) \;=\; \tilde{G}_0^{-1}(\boldsymbol{p}) + \delta m^2 + \Sigma(\boldsymbol{p})\ ,$$

where the meaning of each of the terms in this expression should be clearly derived. You may quote the rules of perturbation theory without derivation.

Hence show, to one-loop order, that

$$m^2(0,T) \;=\; m^2(\Lambda,T) \;+\; \frac{g}{2}\int^{\Lambda} \frac{d^D p}{(2\pi)^D}\, \frac{1}{\boldsymbol{p}^2 + m^2(0,T)}\ .$$

Show that this result is consistent with the Landau-Ginsburg assumption that $m^2(0,T) \sim (T - T_C)$ only for $D > D_C$, where the value of D_C for an ordinary critical point should be calculated.

END OF PAPER