UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 6 June, 2011 1:30 pm to 4:30 pm

PAPER 39

MATHEMATICS OF OPERATIONAL RESEARCH

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define, for mixed strategies, the *upper value* and *lower value* of a two-person zero-sum game.

(b) Explain how to find the upper and lower values by solving linear programs.

By quoting from the theory of linear programming, explain why they are equal.

(c) Suppose that $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, A is a $m \times n$ real matrix, and all components of A, b, and c are positive. Consider the two-person zero-sum game in which each player has $m + n + 1$ pure strategies and the payoff matrix is

$$M = \left(\begin{array}{c|c|c} 0 & A & -b \\ \hline -A^\top & 0 & c \\ \hline b^\top & -c^\top & 0 \end{array}\right).$$

The first (second) diagonal block is a $m \times m$ ($n \times n$) square matrix of zeros. What are the upper and lower values of this game?

(d) Suppose that (for both players) an optimal mixed strategy in the zero-sum game with payoff matix M is $\pi^\top = (p_1, \ldots, p_m, q_1, \ldots, q_n, r)$. Prove that $r \neq 0$.

(e) Explain how to find from π an optimal solution to the linear program $\text{LP} = \{\text{maximize}\, c^\top y : Ay \leqslant b,\, y \geqslant 0\}$.

2

(a) Suppose $X \subseteq \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}^m$ and $b \in \mathbb{R}^m$. Consider the problem

$$\text{P} : \text{minimize } \; f(x) \text{ s.t. } \; h(x) = b, \; x \in X.$$

Formulate the Lagrangian dual problem, P*.

Show that the optimal value of P* provides a lower bound on the optimal value of P.

(b) Suppose that A is a $m \times n$ real matrix and $b \in \mathbb{R}^m$. Consider problems:

$$\begin{aligned} \text{QP} : &\underset{x,y}{\text{minimize}} \; \tfrac{1}{2}x^\top x \\ &\text{s.t. } \; x \geqslant 0, \, y \geqslant 0 \\ &Ax - y = b \end{aligned} \qquad \qquad \begin{aligned} \text{QP}^* : &\underset{\lambda,\,\mu,\,x}{\text{maximize}} \; b^\top \lambda - \tfrac{1}{2}x^\top x \\ &\text{s.t. } \; \lambda \geqslant 0, \, \mu \geqslant 0, \\ &x = A^\top \lambda + \mu \end{aligned}$$

Show that QP* is the Lagrangian dual problem of QP.

(c) Suppose that x, y, λ, μ are feasible for QP and QP* and such that $\lambda^\top y = 0$ and $x^\top \mu = 0$. Show that these variables provide optimal solutions to QP and QP*.

(d) Find a matrix M (involving A) and vector q (involving b) such that solutions to QP and QP* can be found by solving the linear complementarity problem:

$$\text{LCP} : \; \text{Find } w \geqslant 0, \; z \geqslant 0 \text{ s.t. } \; w - Mz = q \text{ and } w^\top z = 0.$$

3

(a) Suppose that A is a $n \times n$ real matrix in which all components are non-negative and $q^\top = (1, \dots, 1) \in \mathbb{R}^n$. Let

$$S = \{(w, z) : w, z \in \mathbb{R}^n,\ w \geqslant 0,\ z \geqslant 0,\ w + Az = q\}.$$

Explain how Nash equilibria of the two-person bimatrix game in which the payoff matrices for the row and column players are A and $B = A^\top$, respectively, are related to solutions to the linear complementary problem:

$$\textbf{LCP}\text{: Find } (w, z) \in S \text{ such that } w^\top z = 0.$$

(b) Starting from the solution at $(w, z) = (q, 0)$, we wish to find a second solution of **LCP** by using Lemke's algorithm to follow a path through a sequence of points in S, each of which has the property that $i = 1$ is the only index (amongst $\{1, 2, \dots, n\}$) for which $z_i w_i$ might be non-zero. Arranging your calculations in a tableau, show that with data

$$A = \begin{pmatrix} 3 & 3 & 0 \\ 4 & 0 & 1 \\ 0 & 4 & 5 \end{pmatrix}$$

the path terminates with tableau

w_1	w_2	w_3	z_1	z_2	z_3	
1/3	−1/4	0	0	1	−1/4	1/12
0	1/4	0	1	0	1/4	1/4
−4/3	1	1	0	0	6	2/3

(c) What happens when the choice $i = 1$ is replaced with $i = 3$?

(d) Suppose that for all $(w, z) \in S$ the total number of non-zero components in w and z is at least n. Prove that the number of solutions of **LCP** is even.

(e) Show that there are solutions to this **LCP** that cannot be found by following some Lemke-algorithm path that starts at $(w, z) = (q, 0)$.

4

(a) Given a graph $G = (V, E)$ and a partition of vertices into nonempty sets S and $\bar{S} = V \setminus S$, define the cut value $C(S, \bar{S})$ as the number of edges having one vertex in S and one vertex in $\bar{S}$. In the MIN-CUT decision problem we are given a graph G and integer k and asked if there exists a cut with $C(S, \bar{S}) \leqslant k$. By using what you know about the Ford-Fulkerson algorithm prove that MIN-CUT is in complexity class P.

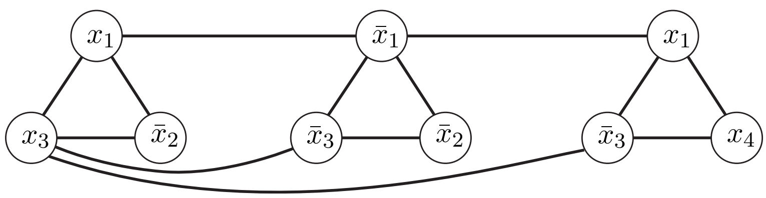
(b) Let MAX-CUT be the problem of finding $\text{OPT}(G) := \max_{S \subset V} C(S, \bar{S})$. Explain how to formulate MAX-CUT as a quadratic programming problem in variables confined to the values 1 and -1.

(c) Consider a Boolean expression, B, that is the conjunction of m clauses, each of which is the disjunction of 3 literals. For example, with $m = 3$ clauses, and $\bar{x}_i =$ 'not x_i',

$$(x_3 \text{ or } x_1 \text{ or } \bar{x}_2) \text{ and } (\bar{x}_3 \text{ or } \bar{x}_1 \text{ or } \bar{x}_2) \text{ and } (\bar{x}_3 \text{ or } x_1 \text{ or } x_4).$$

The NAE-3SAT decision problem asks if it is possible to assign values to the variables (true or false) so that the Boolean expression is true, and also so that the 3 literals in each clause are *not all equal* (i.e. not all true). In the example above, the answer is yes, by taking $x_1 = x_2 = \bar{x}_3 = \bar{x}_4 = \text{true}$.

Let us construct a graph, $H(B)$, in which $3m$ vertices represent the $3m$ literals. Place an edge between any two vertices that represent literals that cannot be equal (such as x_i and $\bar{x}_i$). Suppose this creates K edges. Also place edges between vertices that represent literals in the same clause (giving another $3m$ edges). For the example above, we would have the graph



Use this construction to show that if NAE-3SAT is NP-complete then MAX-CUT is NP-hard. Hint: consider the question: is $\text{OPT}(H(B)) \geqslant K + 2m$?

(d) Consider the following approximation algorithm for MAX-CUT.

Step 1. Arbitrarily partition the vertices into two nonempty sets S and $\bar{S}$.

Step 2. Look for a vertex which if moved from its set to the other set will increase the value of the cut. If no such vertex exists then stop. Otherwise, move this vertex to the other set, and then repeat Step 2.

Let $A(G)$ denote the value of the cut created by this algorithm.
Show that $A(G) \geqslant (1/2)\text{OPT}(G)$. Hint: $\text{OPT}(G) \leqslant |E|$.

5

(a) Let $G = (V, E)$ be a graph with vertex set $V = \{0, \ldots, n\}$. Suppose there is an edge between every pair of vertices, and each edge e has an associated cost $\ell(e)$. Given an edge e, suppose that V can be partitioned into disjoint sets, U and $V \setminus U$, so that e is an edge of least cost between them. Prove that there exists a minimum cost spanning tree that includes e.

(b) Prove that a minimum cost spanning tree can be found by the algorithm which starts with all edges of G coloured white, and at each of n successive steps recolours one edge black, choosing this edge as one of least cost amongst those white edges that could be made black without creating a black cycle.

(c) Suppose that every $\ell(e)$ is a non-negative integer no greater than k. Let S be a nonempty subset of $N = \{1, \ldots, n\}$. As a function of ℓ, let $c(S, \ell)$ denote the least cost of subtree of G which has $|S|$ edges and connects all vertices in S to vertex 0.

For each $j = 1, \ldots k$, define $\ell_j : E \to \{0, 1\}$ by

$$\ell_j(e) = \begin{cases} 0 & \text{if } \ell(e) < j, \\ 1 & \text{if } \ell(e) \geqslant j. \end{cases}$$

Show that $c(S, \ell) = c(S, \ell_1) + \cdots + c(S, \ell_k)$.

(d) In the *minimum cost spanning tree game* the set of players is $N = \{1, \ldots, n\}$ and the characteristic function is defined as $v(S) = c(S, \ell)$, $S \subseteq N$. It is desired to specify a cost sharing, $\{x_{S,i},\ i \in S\}$, for each subset S, having the desirable properties that

$$\sum_{i \in S} x_{S,i} = v(S) \text{ for all } S \subseteq N,\ S \neq \emptyset\,, \tag{1}$$

$$x_{S,i} \geqslant x_{T,i} \text{ for all } i \in S \subset T \subseteq N. \tag{2}$$

Explain why these properties are desirable.

Show that if such numbers exist then $(x_{N,1}, \ldots, x_{N,n})$ is in the core of the game.

(e) Consider a simple case of the above, in which $\ell(e) \in \{0, 1\}$ for all e. Let $S \subseteq N$. For each $i \in S$, set $x_{S,i} = 0$ if, for some $j \in S \cup \{0\}$ and $j < i$, vertex i can be connected to j by a path of cost 0 passing through only vertices in $S \cup \{0\}$. Otherwise set $x_{S,i} = 1$. Prove that with this definition (1) and (2) hold.

How could you solve (1)–(2) in a case that $\ell(e) \in \{0, \ldots, k\}$ for all $e \in E$?

6

(a) A item is to be auctioned between two identical risk-neutral bidders, who have private valuations for winning the item that are independently distributed uniformly on $[0, 1]$. The auction design specifies that the item is won by the highest bidder, and, as functions of their bids, the bidders shall make certain non-negative payments to the auctioneer. All the above (apart from the private valuations) is public knowledge.

Suppose that in equilibrium the optimal strategy of bidder i, when having private valuation v_i, is to participate in the auction and bid $b(v_i)$ if $v_i \geqslant \bar{v}_i$, but to not participate if $v_i < \bar{v}_i$, where $\bar{v}_i$ is some number in $[0, 1]$. Conditional on v_i, let $\pi(v_i)$ and $e(v_i)$ denote, respectively, bidder i's expected profit and expected payment.

Explain why $\pi(v_i) = v_i^2 - e(v_i)$ for $v_i \geqslant \bar{v}_i$, and $\pi(0) = 0$ for $v_i < \bar{v}_i$. Show that

$$\pi(v_i) = (1/2)(v_i^2 - \bar{v}_i^2), \quad v_i \geqslant \bar{v}_i.$$

(b) Consider three auctions designs in which: (i) the winner pays his bid, (ii) the loser pays his bid, (iii) both winner and loser pay their bids. Why in these designs is $\bar{v}_i = 0$?

Find $b(v_i)$ in each case. Verify that in (ii) $b(v_i) \to \infty$ as $v_i \to 1$.

Verify that (i) and (iii) guarantee the same expected revenue for the auctioneer.

(c) Show that if we add to auction (iii) the rule that the minimum permitted bid is $1/4$ then $\bar{v}_1 = \bar{v}_2 = 1/2$.

Show that the expected revenue obtained by the auctioneer in this auction exceeds that in any of (i), (ii) and (iii).

END OF PAPER