UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 13 June, 2011 1:30 pm to 4:30 pm

PAPER 38

STOCHASTIC CALCULUS AND APPLICATIONS

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let W be a real Brownian motion and $x_0 > 0$ be a constant, and set

$$X_t = e^{-W_t + t/2}\left(x_0 - \int_0^t e^{W_s - s/2} ds\right).$$

(a) Show that X is the unique strong solution of the SDE

$$dX_t = (X_t - 1)dt - X_t dW_t,\ X_0 = x_0.$$

You may use without proof any theorem on uniqueness of solutions of SDEs as long as it is carefully stated.

(b) Assuming

$$\mathbb{P}\left(\int_0^\infty e^{W_s - s/2} ds = x_0\right) = 0,$$

use the Brownian law of large numbers to show $\mathbb{P}(X_t \to \infty \text{ or } X_t \to -\infty) = 1$.

(c) Let F be the bounded function

$$F(x) = \begin{cases} e^{-2/x} & \text{if } x > 0 \\ 0 & \text{if } x \leqslant 0 \end{cases}$$

Show $(x-1)F'(x) + \frac{x^2}{2}F''(x) = 0$ for all x, and conclude that $F(X)$ is a local martingale. Briefly explain why $F(X)$ is a true martingale. [You may use the fact that F is infinitely differentiable without proof.]

(d) Prove Dufresne's theorem:

$$\mathbb{P}\left(\int_0^\infty e^{W_s - s/2} ds < x_0\right) = e^{-2/x_0}.$$

2

Let X be a weak solution of the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

and let

$$T = \inf\{t \geqslant 0 : |X_t| > 1\}$$

be the first time X exits the interval $[-1, 1]$. Finally, let $f : [-1, 1] \to [0, \infty)$ be a given function and let V be continuous on $[-1, 1]$ and twice continuously differentiable on $(-1, 1)$ such that

$$b(x)V'(x) + \frac{1}{2}\sigma(x)^2 V''(x) + f(x) = 0 \text{ if } |x| < 1$$

and $V(x) = 0$ if $|x| = 1$.

(a) Show that $M_t = V(X_{t\wedge T}) + \int_0^{t\wedge T} f(X_s)ds$ is a local martingale.

(b) Now suppose that for every $t \geqslant 0$,

$$\mathbb{E}[\sup_{0\leqslant s\leqslant t} f(X_s)] < \infty.$$

Show that M is a true martingale.

(c) Suppose $T < \infty$ a.s. Use the bounded and monotone convergence theorems to conclude that

$$V(x) = \mathbb{E}\left[\int_0^T f(X_s)ds | X_0 = x\right]$$

(d) Now consider the case where $b(x) = b \neq 0$ and $\sigma(x) = \sqrt{2}$ are constants. Show that

$$\mathbb{E}[T|X_0 = x] = \frac{1}{b}\left[-x + \frac{e^b + e^{-b} - 2e^{-bx}}{e^b - e^{-b}}\right].$$

What is the formula when $b = 0$? [You may use without proof the fact that $T < \infty$ a.s. in this case.]

3

Let M be a continuous local martingale with $M_0 = 0$, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geqslant 0})$. Let Z be the local martingale defined by

$$Z_t = e^{M_t - [M]_t/2}.$$

(a) Let X be another continuous local martingale, and let

$$Y = X - [M, X].$$

Show that the process ZY is yet another local martingale.

(b) Suppose Z is a true martingale. Show that ZY^{τ_n} is a true martingale for each n, where $\tau_n = \inf\{t \geqslant 0 : |Y_t - Y_0| > n\}$.

(c) Fix a non-random time horizon $T > 0$, and define an equivalent measure $\mathbb{Q} \sim \mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ by the density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$$

Show that $(Y_t)_{t \in [0,T]}$ is a local martingale under $\mathbb{Q}$.

(d) Now suppose X is a Brownian motion for $\mathbb{P}$. Show that in this case Y is a Brownian motion for $\mathbb{Q}$. You may use any characterisation of Brownian motion you know, provided it is clearly stated.

4

(a) Let X be a continuous, non-negative local martingale such that $X_0 = 1$ and $X_t \to 0$ a.s. as $t \to \infty$. For each $a > 1$, let $\tau_a = \inf\{t \geqslant 0 : X_t > a\}$. Show that

$$\mathbb{P}(\tau_a < \infty) = \mathbb{P}(\sup_{t \geqslant 0} X_t > a) = 1/a.$$

[Hint: compute the expected value of $X_{t \wedge \tau_a} = a\mathbf{1}_{\{\tau_a \leqslant t\}} + X_t \mathbf{1}_{\{\tau_a > t\}}$.]

Let M be a continuous local martingale with $M_0 = 0$ and $[M]_\infty = \infty$ a.s.

(b) State the Dambis–Dubins–Schwarz theorem in terms of M. Give a proof of this theorem under the additional assumption that $t \to [M]_t(\omega)$ is strictly increasing for each ω.

(c) Use the Brownian law of large numbers to show that $M_t - [M]_t/2 \to -\infty$.

(d) Show that

$$\mathbb{P}(\sup_{t \geqslant 0} M_t - [M]_t/2 > y) = e^{-y}$$

for all $y > 0$. [Hint: consider the local martingale $X_t = e^{M_t - [M]_t/2}$.]

5

Let W be a standard Brownian motion, and let f be a bounded function with a continuous, bounded derivative f'. Let $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ denote the density of a $N(0,1)$ random variable.

(a) Let

$$U(t,x) = \int_{-\infty}^{\infty} f(x + \sqrt{1-t}y)\phi(y)dy.$$

By directly computing conditional expectations and using the independence of the increments of Brownian motion, show the process $(U(t,W_t))_{t\in[0,1]}$ is a martingale with respect to the filtration generated by W.

(b) Use Itô's formula to prove that

$$f(W_1) = \int_{-\infty}^{\infty} f(y)\phi(y)dy + \int_0^1 \frac{\partial U}{\partial x}(t,W_t)dW_t.$$

[Hint: $f(W_1) = U(1,W_1)$ and $\int_{-\infty}^{\infty} f(y)\phi(y)dy = \mathbb{E}[f(W_1)] = U(0,0)$.]

(c) Use Itô's isometry to establish the identity

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)^2\phi(x)dx &= \left(\int_{-\infty}^{\infty} f(y)\phi(y)dy\right)^2 \\ &\quad + \int_0^1 \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f'(\sqrt{t}x + \sqrt{1-t}y)\phi(y)dy\right)^2 \phi(x)dx\ dt \end{aligned}$$

[You may use the formula

$$\frac{\partial}{\partial x}U(t,x) = \int_{-\infty}^{\infty} f'(x + \sqrt{1-t}y)\phi(y)dy$$

without proof.]

(d) Apply the Cauchy–Schwarz inequality to establish Poincaré's inequality:

$$\int_{-\infty}^{\infty} f(x)^2\phi(x)dx \leqslant \left(\int_{-\infty}^{\infty} f(x)\phi(s)dx\right)^2 + \int_{-\infty}^{\infty} f'(x)^2\phi(x)dx.$$

[Hint: if X and Y are independent $N(0,1)$ random variables, then $\sqrt{t}X + \sqrt{1-t}Y \sim N(0,1)$.]

6

(a) Fix a non-random time horizon $T > 0$, and let $\mathcal{A}$ be the set of continuous processes $X = (X_t)_{t\in[0,T]}$ such that $|||X||| < \infty$ where

$$|||X|||^2 = \mathbb{E}[\sup_{t\in[0,T]} X_t^2].$$

Show that $\mathcal{A}$ is complete with respect to $|||\cdot|||$, in the sense that if $(X^n)_n$ is a sequence in $\mathcal{A}$ such that

$$|||X^n - X^m||| \to 0$$

as $m, n \to \infty$, then there exists a process $X \in \mathcal{A}$ such that $|||X^n - X||| \to 0$.

(b) Let $(M^n)_n$ be a sequence of continuous martingales such that the sequence $(M_T^n)_n$ of random variables is Cauchy in $L^2(\Omega)$. Use Doob's maximal inequality to show that there exists a process $X \in \mathcal{A}$ such that $|||M^n - X||| \to 0$.

(c) Let $\mathcal{S}$ be the set of processes $H = (H_t)_{t\in[0,T]}$ of the form

$$H = \sum_{n=1}^{N} \mathbf{1}_{(t_{n-1},t_n]} h_n$$

where $0 \leqslant t_0 < \ldots < t_N$ are not random, and for each n the random variable h_n is bounded and $\mathcal{F}_{t_{n-1}}$ measurable. How is the stochastic integral $(H \cdot M)_t = \int_0^t H_s dM_s$ defined when M is a continuous martingale in $\mathcal{A}$ and the process H is in $\mathcal{S}$? Show that $(H \cdot M)$ is a continuous martingale and prove Itô's isometry:

$$\mathbb{E}[(H \cdot M)_T^2] = \mathbb{E} \int_0^T H_s^2 d[M]_s.$$

[You may use the fact that $M^2 - [M]$ is a martingale.]

(d) Let $(H^n)_n$ be a Cauchy sequence in $\mathcal{S}$ with respect to the norm

$$\|H\|^2 = \mathbb{E} \int_0^T H_s^2 d[M]_s.$$

Show that there exists a process $X \in \mathcal{A}$ such that $|||(H^n \cdot M) - X||| \to 0$.

END OF PAPER