UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2011 9:00 am to 12:00 pm

PAPER 35

ADVANCED FINANCIAL MODELS

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $Z \sim N(0,1)$ be a standard normal random variable, and define a function F by the formula

$$F(v,m) = \mathbb{E}[(e^{-v/2+\sqrt{v}Z} - m)^+].$$

(a) Let W be a Brownian motion, let r, σ and S_0 be real constants with $S_0 > 0$, and suppose

$$dS_t = S_t(r\ dt + \sigma dW_t).$$

Show that

$$\mathbb{E}[e^{-rT}(S_T - K)^+] = S_0 F(T\sigma^2, Ke^{-rT}/S_0).$$

for all $K > 0$.

(b) Now, let W and Z be independent Brownian motions, let r, σ_0, ρ and S_0 be real constants with $S_0 > 0$ and $-1 \leqslant \rho \leqslant 1$, and let a and b be smooth functions. Suppose that

$$\begin{aligned} dS_t &= S_t(r\ dt + \sigma_t[\rho dW_t + \sqrt{1-\rho^2}dZ_t]) \\ d\sigma_t &= a(\sigma_t)dt + b(\sigma_t)dW_t. \end{aligned}$$

Assume that the process $\sigma = (\sigma_t)_{t \geqslant 0}$ is bounded and adapted to the filtration generated by W. By conditioning on W, show that

$$\mathbb{E}[e^{-rT}(S_T - K)^+] = \mathbb{E}\left[S_0 \mathcal{E}_T F\left(\int_0^T (1-\rho^2)\sigma_t^2 dt, Ke^{-rT}/(S_0\mathcal{E}_T)\right)\right]$$

where the function F is defined above, and

$$\mathcal{E}_T = \exp\left(-\frac{1}{2}\int_0^T \rho^2\sigma_t^2 dt + \int_0^T \rho\sigma_t dW_t\right)$$

(c) Under the assumptions of part (b), show that

$$\mathbb{E}[e^{-rT}(S_T - K)^+] = \mathbb{E}\left[S_0 F\left(\int_0^T (1-\rho^2)\hat{\sigma}_t^2 dt, Ke^{-rT}\hat{\mathcal{E}}_T/S_0\right)\right]$$

where

$$d\hat{\sigma}_t = (a(\hat{\sigma}_t) + \rho\hat{\sigma}_t b(\hat{\sigma}_t))dt + b(\hat{\sigma}_t)d\hat{W}_t, \quad \hat{\sigma}_0 = \sigma_0.$$

and

$$\hat{\mathcal{E}}_T = \exp\left(-\frac{1}{2}\int_0^T \rho^2\hat{\sigma}_t^2 dt - \int_0^T \rho\hat{\sigma}_t d\hat{W}_t\right)$$

and $\hat{W}$ is a Brownian motion.

2

(a) Let $X = (X_1, \ldots, X_d)$ be a random vector and let

$$C = \{\mathbb{E}(ZX) : Z > 0 \text{ almost surely and } \mathbb{E}(Z|X|) < \infty\} \subseteq \mathbb{R}^d$$

Show that if the set C does not contain 0, then there exists a (non-random) vector $a \in \mathbb{R}^d$ such that $a \cdot X \geqslant 0$ almost surely and $\mathbb{P}(a \cdot X > 0) > 0$.

[You may (but need not) use a version of the separating hyperplane theorem without proof, as long as it is clearly stated. Also, you may (but need not) assume that the random variables $X_1, \ldots, X_d$ are linearly independent in the sense that $a \cdot X = 0$ almost surely implies $a = 0$.]

(b) In the context of a $d+1$ asset, one-period model with a numéraire asset, show that there is no arbitrage only if there exists an equivalent martingale measure. Your answer should contain the definitions of the terms numéraire asset, arbitrage, and equivalent martingale measure.

3

Consider a continuous time market with two assets. Asset 0 is cash with price $B_t = 1$ for all $t \geqslant 0$ and asset 1 is a stock whose price S_t has dynamics

$$dS_t = S_t(\mu_t \, dt + \sigma_t \, dW_t),$$

where W is a Brownian motion and μ and σ are continuous and adapted to the filtration $(\mathcal{F}_t)_{t \geqslant 0}$ generated by W.

(a) What is does it mean to say a trading strategy $(\pi_t)_{0 \leqslant t \leqslant T}$ is admissible?

Suppose $\sigma_t > 0$ for all $t \geqslant 0$ almost surely, and let $\lambda_t = \mu_t / \sigma_t$. Finally, let

$$Z_t = e^{-\frac{1}{2}\int_0^t \lambda_s^2 ds - \int_0^t \lambda_s dW_s}$$

(b) Show that Z is a local martingale.

(c) Show that there exists an admissible strategy π and intial wealth x such that

$$x + \int_0^t \pi_s dS_s = 1/Z_t$$

for all $0 \leqslant t \leqslant T$.

(d) Let M be a positive martingale. Use the martingale representation theorem to show that there exists an admissible strategy π and intial wealth x such that

$$x + \int_0^t \pi_s dS_s = M_t / Z_t$$

for all $0 \leqslant t \leqslant T$.

(e) By considering the martingale $M_t = \mathbb{E}(Z_T | \mathcal{F}_t)$, show that the market has an arbitrage strategy if $\mathbb{E}(Z_T) < Z_0$.

4

Consider a one-period market model with $N+1$ assets: a bond, a stock, and $N-1$ call options. The prices of the bond are $B_0 = 1$ and $B_1 = 1+r$, where r is a constant. The prices of the stock are given by a constant S_0 and a random variable S_1 taking values in $\{0, 1, \dots, N-1, N\}$ for a given integer $N \geqslant 4$. Finally, the time-0 price of the call option with strike $K \in \{1, \dots, N-1\}$ is denoted $C(K)$. Assume that there is no arbitrage in the market.

(a) Show that $0 \leqslant S_0 \leqslant \frac{N}{1+r}$.

(b) Show that $\left(S_0 - \frac{K}{1+r}\right)^+ \leqslant C(K) \leqslant S_0$ for all $1 \leqslant K \leqslant N-1$.

(c) Show that $0 \leqslant C(K) - C(K+1) \leqslant \frac{1}{1+r}$ for all $1 \leqslant K \leqslant N-2$.

(d) Show that $C(K+1) - 2C(K) + C(K-1) \geqslant 0$ for all $2 \leqslant K \leqslant N-2$.

(e) Now suppose a contingent claim with time-1 payout $\xi_1 = g(S_1)$ is introduced, where g is the function

$$g(S) = \begin{cases} 1 & \text{if } S = K_0 \\ 0 & \text{otherwise} \end{cases}$$

where $0 \leqslant K_0 \leqslant N$ is a given integer. In each of the following cases, find the unique time-0 price ξ_0 such that the augmented market has no arbitrage:

(i) $2 \leqslant K_0 \leqslant N-2$,

(ii) $K_0 = N-1$,

(iii) $K_0 = 0$.

[You may use a fundamental theorem of asset pricing without proof, as long as it is stated carefully.]

5

Consider a market with two assets, a bank account with time-t price e^{rt} and a stock whose price dynamics satisfy

$$\begin{aligned} dS_t &= S_t(r\ dt + \sqrt{v_t}dW_t) \\ dv_t &= (a - bv_t)dt + c\sqrt{v_t}(\rho dW_t + \sqrt{1-\rho^2}dZ_t) \end{aligned}$$

where r, a, b, c and ρ are contants, with $a, b > 0$ and $-1 \leqslant \rho \leqslant 1$, and W and Z are independent Brownian motions.

Let $F : [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the partial differential equation

$$\frac{\partial F}{\partial t} + Sr\frac{\partial F}{\partial S} + (a - bv_t)\frac{\partial F}{\partial v} + \frac{1}{2}S^2 v\frac{\partial^2 F}{\partial S^2} + c\rho S v\frac{\partial^2 F}{\partial S \partial v} + \frac{1}{2}c^2 v\frac{\partial^2 F}{\partial v^2} = rF$$

with boundary condition $F(T, S, v) = \sqrt{S}$.

Introduce a contingent claim with time-T payout $\xi_T = \sqrt{S_T}$.

(a) Show that there is no arbitrage in the augmented market if the time-t price of the contingent claim is given by $\xi_t = F(t, S_t, v_t)$. You may use a fundamental theorem of asset pricing as long as it is stated carefully.

Suppose that $F(t, S, v) = \sqrt{S}e^{A(t)v + B(t)}$ for some functions $A, B : [0,T] \to \mathbb{R}$.

(b) Show that A satisfies an ordinary differential equation. You should derive the equation, including the boundary conditions, but need not solve it.

(c) Show that the function B is given by

$$B(t) = -(T-t)r/2 + k\int_t^T A(s)ds$$

for a constant k which you should find in terms of the model parameters.

6

Let $Y = (Y_t)_{t\in\{0,\dots,T\}}$ be a bounded process adapted to a filtration $(\mathcal{F}_t)_{t\in\{0,\dots,T\}}$. Let U be the Snell envelope of Y, defined by $U_T = Y_T$ and

$$U_t = \max\{Y_t, \mathbb{E}(U_{t+1}|\mathcal{F}_t)\} \text{ for } 0 \leqslant t < T.$$

(a) Prove Doob's decomposition theorem: there exists martingale M with $M_0 = 0$ and predictable non-decreasing process A with $A_0 = 0$ such that

$$U_t = U_0 + M_t - A_t$$

for all $t = 0, \dots, T$.

(b) Use the optional stopping theorem to show that for every stopping time $\tau \leqslant T$,

$$\mathbb{E}(Y_\tau) \leqslant U_0.$$

(c) Let $\tau^* = \min\{t : A_{t+1} > 0\}$, with the convention that $\tau^* = T$ on the set $\{A_T = 0\}$. Show that

$$\mathbb{E}(Y_{\tau^*}) = U_0$$

(d) Let N be a martingale with $N_0 = 0$. Use the optional stopping theorem to show

$$\mathbb{E}\left[\max_{t\in\{0,\dots,T\}}(Y_t - N_t)\right] \geqslant \sup_{\tau\leqslant T} \mathbb{E}(Y_\tau - N_\tau) = U_0$$

where the supremum is over all stopping times $\tau \leqslant T$.

END OF PAPER