UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 7 June, 2011 1:30 pm to 3:30 pm

PAPER 34

TIME SERIES AND MONTE CARLO INFERENCE

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Time Series

(a) Define the autocovariances γ_k, $k \in \mathbb{Z}$, for a weakly stationary process $\{X_t\}$. Assume that a spectral density function $f(\lambda)$ exists for $\{X_t\}$. Write down an expression for $f(\lambda)$ in terms of the autocovariances. Find the autocovariances and the spectral density function for a white noise process $\{\varepsilon_t\}$ with mean zero and variance σ^2.

(b) Suppose that $\{X_t\}$ satisfies

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \varepsilon_t, \qquad (*)$$

where $\phi_1, \ldots, \phi_p$ are real constants and $\{\varepsilon_t\}$ is as in (a). Write down a condition which implies there is a unique weakly stationary solution to $(*)$. Assume that this condition is satisfied. State the Filter Theorem, and show that $\{X_t\}$ has spectral density function

$$f_X(\lambda) = \frac{\sigma^2}{2\pi |D(e^{i\lambda})|^2},$$

where $D(z)$ is a polynomial that you should specify in terms of the ϕ_js.

(c) Consider the process $Y_t = cY_{t-12} + \varepsilon_t$, where c is a real constant and $\{\varepsilon_t\}$ is as in (a). Find the range of values of c that satisfy the condition for the uniqueness and existence of a weakly stationary solution. Assume that c is in this range of values. Find the spectral density function of $\{Y_t\}$. Use the Yule–Walker equations for the process $\{Y_t\}$ to find its autocorrelation function.

[Results from lectures may be quoted and used without proof.]

2 Time Series

Observations $\{X_t\}$ are said to be in state space form if there are states $\{S_t\}$ such that $X_t = FS_t + v_t$ and $S_t = GS_{t-1} + w_t$, where X_t and S_t may be vector-valued, and where $\{v_t\}$ and $\{w_t\}$ are independent sequences of independent normally distributed vectors with mean zero and with covariance matrices V and W respectively (zero covariance matrices are allowed). Throughout this question, $\{\varepsilon_t\}$ denotes a Gaussian white noise process with variance σ^2.

(a) Consider the MA(1) model $Z_t = \varepsilon_t + \theta\varepsilon_{t-1}$. Show that Z_t can be written in state space form with the state at time t given by $S_t = \begin{pmatrix} \varepsilon_t \\ \varepsilon_{t-1} \end{pmatrix}$, and specify F, G, v_t, w_t, V and W.

(b) Consider an ARMA(1,1) model $X_t = \phi X_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1}$, where $|\phi| < 1$. Show that $\{X_t\}$ can be written in state space form with the state at time t given by $\begin{pmatrix} X_{t-1} \\ \varepsilon_t \\ \varepsilon_{t-1} \end{pmatrix}$.

(c) Consider X_t as in (b). Let B be the backward shift operator defined by $BX_t = X_{t-1}$. Show that $X_t = \theta(B)U_t$ where $\theta(z) = 1 + \theta z$ and where $\phi(B)U_t = \varepsilon_t$ where $\phi(z) = 1 - \phi z$. Hence give an alternative state space formulation for $\{X_t\}$ with the state at time t given by $\begin{pmatrix} U_t \\ U_{t-1} \end{pmatrix}$.

(d) Suppose that $\{Y_t\}$ is an ARIMA(1,1,1) process that satisfies $(I - B)Y_t = X_t$, where I is the identity operator and X_t is as in (b). Show that $Y_t = X_t + Y_{t-1}$, and hence find a state space formulation for $\{Y_t\}$ with state at time t given by $\begin{pmatrix} U_t \\ U_{t-1} \\ Y_{t-1} \end{pmatrix}$, where $\{U_t\}$ is as in (c).

(e) Consider the ARIMA(1,2,1) process R_t satisfying $(I - B)^2 R_t = X_t$ where X_t is as in (b). Find a state space formulations for R_t with the state at time t given by $\begin{pmatrix} U_t \\ U_{t-1} \\ R_{t-1} \\ R_{t-2} \end{pmatrix}$, where $\{U_t\}$ is as in (c).

3 Monte Carlo Inference

Describe the *importance sampling estimator* θ_Q^* for estimating $\theta := \mathbb{E}_P\{\phi(X)\}$ from n independent draws from a proposal distribution Q. Under what conditions is it unbiased?

Assuming these conditions, show that $\mathrm{var}(\theta_Q^*)$ is minimised when $q(x) = q_0(x) := p(x)|\phi(x)|/k$, where p, q are the densities of P, Q respectively, and $k = \mathbb{E}_P\{|\phi(X)|\}$; and that the minimised variance is $n^{-1}(k^2 - \theta^2)$.

Using *e.g.* the Box–Muller method, we generate independent standard normal variables Y, Z. A new variable X is then constructed as

$$X := Y \,\mathrm{sign}(\lambda Y - Z)$$

where $\lambda \in \mathbb{R}$. Let P denote the distribution of X. Show that

$$\begin{aligned} \mathbb{E}(X^2) &= 1 && (1)\\ \mathbb{E}(|X|) &= \sqrt{2/\pi} && (2)\\ \mathbb{E}(X) &= \lambda\sqrt{2/\pi(1+\lambda^2)}. && (3) \end{aligned}$$

[*Hint for* (2)*:* Use $x\phi(x) = -\phi'(x)$ and integration by parts.]
[*Hint for* (3)*:* You may use the identity

$$(1+\lambda^2)X = \lambda\,|\lambda Y - Z| + (Y + \lambda Z)\,\mathrm{sign}(\lambda Y - Z) \qquad (4)$$

and that $(Y + \lambda Z)$ and $(\lambda Y - Z)$ are independent.]

For large λ, compare the performance of the optimal importance sampling estimator of $\mathbb{E}_P(X)$ with that of the sample mean of n independent draws from P.

4 Monte Carlo Inference

Describe the general form of the Metropolis-Hastings (M-H) algorithm for Markov chain Monte Carlo simulation from a distribution P with density $p(x)$. Show that if the initial distribution is P then the chain is reversible, and that P is a stationary distribution of the chain.

An attempt is made to simulate from P by rejection sampling using a proposal distribution Q, having density $q(x)$. However we can not establish an upper bound on $p(x)/q(x)$. Let $u(x) := p(x)/c\,q(x)$, where $c > 0$ is some constant. We repeatedly and independently generate Y from Q, and independent $U \sim \mathrm{Unif}(0,1)$, until $U \leqslant u(Y)$: then return $X = Y$. Show that the density of X is

$$f(x) \propto \min\{p(x), c\,q(x)\}. \tag{1}$$

In order to correct the discrepancy of f from p, the above procedure is used as the basis of a Markov chain Monte Carlo routine, as follows. If the current state is x, a new proposal value y is generated by rejection sampling, as above, from the density f in (1), independently of x. This is then accepted or rejected as the next state, according to the M-H procedure with target density p. Write down the acceptance probability $\alpha(x,y)$, distinguishing the cases that each of $u(x), u(y)$ is or is not less than 1.

What happens when in fact $p(x) \leqslant c\,q(x)$ for all x?

END OF PAPER