UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2011 9:00 am to 11:00 am

PAPER 33

OPTIMAL INVESTMENT

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

An investor may invest his wealth in a riskless asset earning interest at rate r and in a risky asset with growth rate μ and volatility σ. If w_t denotes his wealth at time t, c_t his consumption rate at time t, and θ_t the cash value of his holding of the risky asset at time t, explain briefly why his wealth evolves as

$$dw_t = rw_t\,dt + \theta_t(\sigma dW_t + (\mu - r)dt) - c_t dt.$$

What does it mean to say that a portfolio-consumption pair $(\theta_t, c_t)_{t\geqslant 0}$ is *admissible*? Supposing that his objective is to obtain

$$\sup_{c\geqslant 0,\theta} E\left[\int_0^\infty e^{-\rho t} U(c_t, w_t)\,dt\right],$$

derive the Hamilton-Jacobi-Bellman equation for the value function.

Assuming that

$$U(c, w) = \frac{w^\alpha c^\beta}{1-R}$$

where α, β are negative constants, and $R = 1 - \alpha - \beta$, find a scaling relationship which must be satisfied by the value function V. Hence show that the value function is

$$V(w) = \frac{Aw^{1-R}}{1-R},$$

where

$$A^{1/(\beta-1)} = \left(\frac{\beta}{1-R}\right)^{\beta/(\beta-1)} \frac{R\gamma_M}{1-\beta},$$

and $R\gamma_M = \rho + (R-1)\{r + (\mu - r)^2/(2\sigma^2 R)\}$.

Comment briefly on the form which this takes when $\alpha = 0$.

2

An agent invests in a riskless asset with constant growth rate μ and volatility σ, and in a bank account earning interest at rate r. He consumes from his wealth up until some stopping time τ of his choosing, at which time he leaves the market and goes into retirement, receiving reward $F(w_\tau)$ at that time, where w_t is his wealth at time t, and F is some increasing strictly concave function. His objective is to obtain

$$V(w) = \sup_{c \geqslant 0, \theta, \tau} E\left[\int_0^\tau e^{-\rho t} U(c_t)\, dt \; + e^{-\rho\tau} F(w_\tau) \;\middle|\; w_0 = w \right],$$

where U is an increasingly strictly concave function, and c_t is the consumption rate at time t. Find the HJB equation which must be satisfied by V.

Supposing that $U(x) = x^{1-R_1}/(1-R_1)$ and that $F(x) = x^{1-R_2}/(1-R_2)$ for some $R_2 > R_1 > 1$, explain briefly why you would expect the optimal stopping time to be of the form

$$\tau = \inf\{t : w_t > w^*\}$$

for some constant w^*. Rexpressing the HJB equation in terms of dual variables, obtain the solution of the problem as explicitly as you can.

3

Consider the problem of an agent who can trade a riskless asset bearing a constant interest rate $r \geqslant 0$ and a risky asset with log-Brownian dynamics, so that the wealth w_t of the agent at time t evolves as

$$dw_t = rw_t\, dt + \theta_t(\sigma dW_t + (\mu - r)dt),$$

where θ_t is the amount of wealth he chooses to invest in the risky asset at time t. His objective is to maximize $EU(w_T)$, where U is C^2 strictly increasing, strictly concave, and satisfies the Inada conditions. You may assume that the value function

$$V(t, w) \equiv \sup_{\theta} E[\, U(w_T) \mid w_t = w\,]$$

is finite-valued and C^2 throughout $[0, T] \times (0, \infty)$. Derive the HJB equation satisfied by V.

Suppose that the agent follows the optimal strategy (θ^*_t), generating wealth process (w^*_t). Suppose that at some time $t \in (0, T)$ he is offered the chance to buy himself an additional ε units of a bounded contingent claim Y which he will receive at time T, for which he must pay εp_t at time t. Thinking of ε as a small parameter, by considering and comparing the leading-order impact on his objective at time t of:

(i) receiving an additional εY at time T;

(i) paying εp_t at time t

show that

$$p_t V_w(t, w^*_t) = E[V_w(T, w^*_T)Y \mid \mathcal{F}_t\,]$$

and deduce that $V_w(t, w^*_t) \equiv \xi_t$ must act as a state-price density: for any traded asset S_t, the process $\xi_t S_t$ must be a martingale.

Using the HJB equation satisfied by V show that ξ satisfies the SDE

$$d\xi_t = \xi_t[\, -\kappa dW_t - r dt\,]$$

where $\kappa = (\mu - r)/\sigma$ and comment briefly.

4

An agent may invest in a riskless bank account paying interest at constant rate r, and in a risky stock whose evolution is given by

$$\begin{aligned} dS_t &= S_t(\sigma_t dW_t + \mu dt) \\ d\sigma_t &= q(\sigma_t) dW'_t + b(\sigma_t) dt \end{aligned}$$

where W and W' are two Brownian motions, $dW_t dW'_t = \rho dt$. The agent's goal is to maximize (subject to keeping wealth non-negative at all times) the objective

$$E \int_0^\infty e^{-\alpha t} U(c_t)\, dt,$$

where $\alpha > 0$, $(c_t)_{t \geqslant 0}$ is the chosen consumption process, and U is a CRRA utility with coefficient $R > 1$ of relative risk aversion. Characterize the value function of this problem as completely as you can.

Briefly discuss methods for solving numerically the equations which result, assuming that $\rho = 0$.

Comment briefly on the form taken by the solution when $q \equiv 0$, and $b \equiv 0$.

END OF PAPER