UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 10 June, 2011 1:30 pm to 3:30 pm

PAPER 30

SCHRAMM-LOEWNER EVOLUTIONS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let $(K_t)_{t\geqslant 0}$ be a family of compact $\mathbb{H}$-hulls. Define the local growth property. Explain what is meant by the Loewner transform $(\xi_t)_{t\geqslant 0}$ of $(K_t)_{t\geqslant 0}$, justifying why it is well-defined.

(b) Let $(\xi_t)_{t\geqslant 0}$ be a continuous real-valued path with $\xi_0 = 0$. Explain what is meant by the Loewner flow driven by $(\xi_t)_{t\geqslant 0}$, and explain how to define an expanding family of compact $\mathbb{H}$-hulls $(K_t)_{t\geqslant 0}$ from it. [You are not required to show that K_t is indeed a compact $\mathbb{H}$-hull.]

(c) Let $(K_t)_{t\geqslant 0}$ have the local growth property, and let $(\xi_t)_{t\geqslant 0}$ be its Loewner transform. Show that $(\xi_t)_{t\geqslant 0}$ is continuous.

Assume that $\operatorname{hcap}(K_t) = 2t$ for all $t \geqslant 0$. Let g_t be conformal isomorphism from $H_t = \mathbb{H} \setminus K_t$ onto $\mathbb{H}$ such that $g_t(z) - z \to 0$ as $z \to \infty$. Show that $(g_t)_{t\geqslant 0}$ is the Loewner flow driven by $(\xi_t)_{t\geqslant 0}$.

[You may use without proof the following estimates on conformal maps: if K is a compact $\mathbb{H}$-hull and $K \subset \xi + r\mathbb{D}$ for some $\xi \in \mathbb{R}, r > 0$, then

$$|g_K(z) - z| \leqslant 3r$$

for all $z \in H = \mathbb{H} \setminus K$, and

$$|g_K(z) - z - \frac{\operatorname{hcap}(K)}{z - \xi}| \leqslant \frac{Cr \operatorname{hcap}(K)^2}{(z - \xi)^2}$$

for all $z \in H$ such that $|z - \xi| \geqslant 2r$. Here g_K denotes the conformal isomorphism from H to $\mathbb{H}$ such that $g_K(z) - z \to 0$ as $z \to \infty$.]

2

(a) Let $\alpha \in (0, \pi)$. Let $\gamma(t)$ be the curve in $\bar{\mathbb{H}}$ defined by $\gamma(t) = r(t)e^{i\alpha}$, where $r(t) \geqslant 0$ is chosen so that if $K_t = \gamma((0, t])$, then $\text{hcap}(K_t) = 2t$. Let $(\xi_t)_{t \geqslant 0}$ be the Loewner transform of $(K_t)_{t \geqslant 0}$. Show that for all $\lambda > 0$, $g_{\lambda^2 t}(z) = \lambda g_t(z/\lambda)$. Deduce that there exists a (necessarily unique) $C_\alpha \in \mathbb{R}$ such that $\xi_t = C_\alpha \sqrt{t}$ for all $t \geqslant 0$.

(b) Let $z = iy$ with $y > 1$, and let B be a Brownian motion starting from z. Choose $t > 0$ such that $|\gamma(t)| = 1$, let $K = K_t$ and set $H = \mathbb{H} \setminus K$ and $T = \inf\{t \geqslant 0 : B_t \notin H\}$. Define ∂_- and ∂_+ as follows. Say that $B_T \in \partial_-$ if there exists a sequence $t_n < T$ with $t_n \uparrow T$ such that $\arg(B_{t_n}) > \alpha$, where if $z \in \mathbb{C}$ then $\arg(z) \in (-\pi, \pi]$ denotes the argument of z. Otherwise, say that $B_T \in \partial_+$. Argue that if $\alpha \leqslant \pi/2$, then $\mathbb{P}(B_T \in \partial_-) \geqslant \mathbb{P}(B_T \in \partial_+)$.

[Hint: let $I = [0, \gamma(t)]$ and let I^* denote the reflection of I with respect to the imaginary axis. Let τ be the hitting time of the unit circle C, and assume $\tau < T$. Consider the random time S defined as the last time that the Brownian motion B hits C before touching $I \cup I^* \cup \mathbb{R}$, and observe that the law of B_S is symmetric with respect to the imaginary axis.]

(c) Let g_K be conformal isomorphism from H onto $\mathbb{H}$ such that $g_K(z) - z \to 0$ as $z \to \infty$. Taking $K = K_t$ where t is chosen as above (i.e., so that $|\gamma(t)| = 1$), draw a picture to represent the effect of the transformation g_K, highlighting in particular where different pieces of the boundary of H are mapped (you are not expected to justify that this picture is correct).

Deduce from (b) and conformal invariance of Brownian motion that $C_\alpha \geqslant 0$ if and only if $\alpha \leqslant \pi/2$.

3

Let $\kappa \geqslant 0$ and let g_t be the Loewner flow associated with an $SLE(\kappa)$ curve. Let K_t denote the corresponding hull and $H_t = \mathbb{H} \setminus K_t$, where $\mathbb{H}$ is the complex upper-half plane. Let $\zeta(z) = \inf\{t \geqslant 0 : z \in K_t\}$ (for $z \in \bar{\mathbb{H}} \setminus \{0\}$), and let $\xi_t = -\sqrt{\kappa} B_t$ be the associated Loewner transform (hence B is a standard Brownian motion).

(a) State the Loewner differential equation, and deduce that if $x \in \mathbb{R} \setminus \{0\}$, then for all $t < \zeta(x)$,

$$\frac{d}{dt} g_t'(x) = -\frac{2 g_t'(x)}{(g_t(x) - \xi_t)^2}.$$

You may exchange the order of differentiation without justification.

(b) Justify briefly that $g_t'(x) > 0$ for $t < \zeta(x)$. Let $v_t = \log g_t'(x)$. Find an equation for dv_t/dt, and deduce that for all $b > 0$,

$$g_t'(x)^b = \exp\left(-ab \int_0^t \frac{1}{X_s(x)^2} ds\right), \quad t < \zeta(x),$$

where $a = 2/\kappa$ and $X_t(x) = (g_t(x) - \xi_t)/\sqrt{\kappa}$. Explain why this implies that for each fixed t, $g_t'(x)$ is increasing in $x \in (r_t, \infty)$, where $r_t = \sup\{\bar{K}_t \cap \mathbb{R}\}$.

(c) Assume that $x \in (0, \sqrt{\kappa})$. Show that $dX_t(x) = dB_t + (a/X_t(x))dt$ for $t < \zeta(x)$ and $X_0(x) = x/\sqrt{\kappa} \in (0,1)$. Set $\sigma = \inf\{t \geqslant 0 : X_t(x) \in \{0,1\}\}$. Let $Z_t = \exp(-ab \int_0^{t \wedge \zeta(x)} ds/X_s(x)^2)$, and let $\phi(x) = \mathbb{E}(Z_\sigma)$. Check that

$$\mathbb{E}(Z_\sigma | \mathcal{F}_t) = \phi(X_{t \wedge \sigma}) Z_{t \wedge \sigma},$$

where $(\mathcal{F}_t)_{t \geqslant 0}$ is the filtration generated by $(B_t)_{t \geqslant 0}$. Deduce that

$$\phi''(x) + \frac{2a}{x}\phi'(x) - \frac{2ab}{x^2}\phi(x) = 0.$$

[You may use without proof that ϕ is twice continuously differentiable.]

4

(a) Define the notion of chordal $SLE(\kappa)$ in a domain D from a boundary point z_0 to a boundary point z_1, taking care to explain to what extent it is uniquely specified. Explain what is meant by the locality property.

(b) Show that $SLE(6)$ has the locality property. In your proof you may use without proof the fact that if $\Phi : N \to N^*$ is a conformal isomorphism from a neighbourhood N in $\mathbb{H}$ of a segment of the real line I to another neighbourhood N^* of a segment $I^* \subset \mathbb{R}$, then $\text{hcap}(K_t^*) = 2\int_0^t \Phi_t'(\xi_s)^2 ds$ and $\dot{\Phi}_t(\xi_t) = -3\Phi_t''(\xi_t)$. Here $(K_t)_{t\geqslant 0}$ is a family of compact $\mathbb{H}$-hulls with local growth, parametrized so that $\text{hcap}(K_t) = 2t$, with Loewner transform ξ_t. $K_t^* = \Phi(K_t)$, $\Phi_t = g_t^* \circ \Phi \circ g_t^{-1}$ with $g_t = g_{K_t}$ and $g_t^* = g_{K_t^*}$, and these identities are true for $t < T = \inf\{t \geqslant 0 : K_t \notin N \cup (\partial N \cap \mathbb{R})\}$.

(c) Let Δ denote the equilateral triangle with vertices $a = 0$, $b = 1$, $c = e^{i\pi/3}$. Let ϕ denote a map from $\mathbb{H}$ to Δ such that $\phi(0) = a$, $\phi(1) = b$, $\phi(\infty) = c$. Why does ϕ exist? Is it uniquely defined? Explain why for $x > 1$, $\phi(x) = 1 + e^{2i\pi/3}\phi((x-1)/x)$. [Hint: consider the conformal maps $z \mapsto f(z) = 1/(1-z)$ and $z \mapsto g(z) = 1 + e^{2i\pi/3}z$.]

(d) Let γ be an $SLE(6)$ curve in Δ from 0 to 1. Let $t^* = \inf\{t \geqslant 0 : \gamma_t \in [b, c]\}$. Why is t^* well-defined? Show that γ_{t^*} is uniformly distributed on the segment $[b, c]$. [You may use without proof results about hitting probabilities of $SLE(\kappa)$ on the real line, provided that the result is clearly stated. You may also use without proof that the map $\phi(z)$ from part (c) has the expression

$$\phi(z) = c\int_0^z \frac{dw}{w^{2/3}(1-w)^{2/3}},$$

where c is chosen so that $\phi(1) = 1$.]

END OF PAPER