UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 7 June, 2011 1:30 pm to 4:30 pm

PAPER 3

COMMUTATIVE ALGEBRA

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Recall that in this course all rings are commutative.

1

Let A be a ring. What does it mean to *localise* at a prime ideal P of A? Prove that such a localisation A_P exists for every prime ideal P. Explain in what sense A_P is unique.

Suppose that for every prime ideal P of A the ring A_P has no nilpotent elements except 0. Can A have a non-zero nilpotent element?

Suppose that for every prime ideal P of A the ring A_P is an integral domain. Must A be an integral domain?

2

What does it mean to say that a ring is *Noetherian*?

Which of the following are true? Give proofs or counter-examples as appropriate.

(a) A subring of a Noetherian ring is Noetherian.

(b) A quotient of a Noetherian ring is Noetherian.

(c) If A is a Noetherian ring then the formal power series ring $A[[x]]$ is Noetherian.

(d) If A is a Noetherian ring then every Artinian A-module is Noetherian.

(e) If A is a Noetherian ring then every Noetherian A-module is Artinian.

3

Let A be a ring. Show that the following are equivalent:

(i) Every prime ideal of A is an intersection of maximal ideals;

(ii) For every ideal I of A, $N(A/I) = \mathrm{Jac}(A/I)$;

(iii) Every prime ideal of A which is not maximal is the intersection of the prime ideals strictly containing it.

Show that every finitely generated $\mathbb{R}$-algebra satisfies the conditions above.

4

Let A be an integral domain.

What does it mean to say that an A-module is *flat*? What does it mean to say that an A-module is *projective*. Show that all projective A-modules are flat.

Recall that an A-module is *torsion-free* if for all $a \in A\backslash 0$ and all $m \in M\backslash 0$, $am \neq 0$. Show that every A-module has a largest torsion-free quotient.

Show that all flat modules are torsion free.

5

Suppose that A is an integral domain. What does it mean to say that A is *integrally closed*? What does it mean to say that A is a *valuation ring*?

Prove that the intersection of all valuation rings that lie between A and the field of fraction of A is the unique smallest integrally closed ring containing A.

6

Suppose that A is a filtered ring with a Noetherian Rees ring. What does it mean to say that an A-module M has a *good filtration*? What is the *completion* of M?

Show that if

$$0 \to L \to M \to N \to 0$$

is a short exact sequence of A-modules with good filtrations then there is an induced short exact sequence

$$0 \to \widehat{L} \to \widehat{M} \to \widehat{N} \to 0.$$

END OF PAPER