UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 8 June, 2011 1:30 pm to 3:30 pm

PAPER 29

PERCOLATION AND RELATED TOPICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider bond percolation on the square lattice with edge-density $p = \frac{1}{2}$. Let $A_{m,n}$ be the event that there exists an open path traversing the rectangle $[0, m] \times [0, n]$ from left to right.

(a) Prove that there exist strictly positive constants α_1, α_2 such that: for all $m \geqslant 1$,

$$\mathbb{P}(A_{m,m}) > \alpha_1, \qquad \mathbb{P}(A_{2m,m}) > \alpha_2.$$

(b) Deduce that, for any $r = 2, 3, \ldots$, there exists $\alpha_r > 0$ such that: for all $m \geqslant 1$, $\mathbb{P}(A_{rm,m}) > \alpha_r$.

(c) Let $\rho \geqslant 1$, and let R be a rectangle of $\mathbb{R}^2$ with side-lengths a, ρa. [You may not assume that the sides of R are parallel to the axes.] Let A_R be the event that there exists an open path of the lattice containing a sub-path traversing R between its two short sides. Outline a proof that there exists a constant $\alpha_\rho > 0$, independent of a and R, such that $\mathbb{P}(A_R) > \alpha_\rho$.

2 Consider bond percolation on the d-dimensional cubic lattice $\mathbb{Z}^d$.

(a) Explain how to construct an increasing family $(\eta_p : p \in [0, 1])$ of processes such that η_p is a bond percolation process with density p.

(b) Define the critical probability p_c, and state the uniqueness theorem for the number of infinite open clusters in a percolation process.

(c) Let I_p be the event that the origin lies in some infinite open path in η_p. Show that I_p is increasing in p. Deduce that the percolation probability $\theta(p) = \mathbb{P}(I_p)$ satisfies

$$\theta(p) - \theta(p-) = \mathbb{P}\left(I_p \cap \left\{ \bigcap_{p' < p} \overline{I_{p'}} \right\} \right),$$

where $\theta(p-) = \lim_{p' \uparrow p} \theta(p')$ and $\overline{I_{p'}}$ is the complement of $I_{p'}$.

(d) Deduce that θ is a left-continuous function on the interval $(p_c, 1]$.

3

(a) Let μ be a probability measure on the product space $\Sigma = \{-1,1\}^V$, where V is a finite set. State the FKG inequality, including a careful formulation of the FKG lattice condition subject to which it is valid.

(b) Let $\Lambda_n = [-n,n]^d$, viewed as the vertex-set of a finite subgraph of the d-dimensional cubic lattice $\mathbb{Z}^d$. The Ising measure on Λ_n is the probability measure on $\Sigma_n = \{-1,1\}^{\Lambda_n}$ given by

$$\phi_n(\sigma) = \frac{1}{Z} e^{-H(\sigma)}, \qquad H(\sigma) = -\beta \sum_{u \sim v} \sigma_u \sigma_v - h \sum_u \sigma_u,$$

for $\sigma = (\sigma_u : u \in \Lambda_n) \in \Sigma_n$. The first (resp., second) summation is over all nearest-neighbour pairs $u, v \in \Lambda_n$ (resp., all $u \in \Lambda_n$), and the constant Z is chosen appropriately.

(i) Under what conditions on β, h does ϕ_n satisfy the FKG inequality.

(ii) Write $\partial\Lambda_n = \Lambda_n \setminus \Lambda_{n-1}$ for the boundary of the box, and let

$$B_n = \big\{\sigma_v = 1 \text{ for all } v \in \partial\Lambda_n\big\}.$$

Let $\beta \geqslant 0$, and let A be an increasing event in the infinite sample space $\Sigma = \{-1,1\}^{\mathbb{Z}^d}$ that is defined in terms of the states of only finitely many vertices. Show that, for all large n, $\phi_n(A \mid B_n)$ is decreasing in n, and deduce that the limit $\phi^1(A) = \lim_{n\to\infty} \phi_n(A \mid B_n)$ exists.

(iii) Explain briefly why ϕ^1 may be extended to a probability measure on Σ.

4

Write an essay on Cardy's formula. Your essay should include a clear statement of the main theorem. You should include outlines of the main steps of the proof, but need not give full details of their justification.

END OF PAPER