UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 6 June, 2011 9:00 am to 12:00 pm

PAPER 28

ADVANCED PROBABILITY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

All the results of the course may be used, unless stated otherwise.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

a) Let $(X_t)_{t \geqslant 0}$ be a Lévy process constructed on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that, for any $t_0 > 0$, the process $(X_{t_0+t} - X_{t_0})_{t \geqslant 0}$ is a Lévy process independent under $\mathbb{P}$ of $\sigma(X_s\,;\, s \leqslant t_0)$.

b) State and prove the strong Markov property for Brownian motion. Does your proof work for any Lévy process (justify your answer)?

c) Let A be a Borel subset of $\mathbb{R}_+$, $x > 0$ and $t > 0$. Denote by $\mathbb{P}_x$ (resp. $\mathbb{P}_{-x}$) the law of a Brownian motion started from x (resp. $-x$). Prove that

$$\mathbb{P}_x(B_s > 0 \text{ for all } 0 \leqslant s \leqslant t, \text{ and } B_t \in A) = \mathbb{P}_x(B_t \in A) - \mathbb{P}_{-x}(B_t \in A).$$

2

Let $(X_n)_{n \geqslant 0}$ be a sub-martingale defined on some filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geqslant 0}, \mathbb{P}\big)$. Recall that a process $(A_n)_{n \geqslant 0}$ is said to be previsible if each A_n is integrable and $\mathcal{F}_{n-1}$-measurable.

a) Prove that (up to modification on a set of null probability) there exists a unique martingale $(M_n)_{n \geqslant 0}$ null at 0 and a unique increasing previsible process $(A_n)_{n \geqslant 0}$ such that we have almost surely $X_n = X_0 + M_n + A_n$, for all $n \geqslant 0$.

b) Suppose $X_n \geqslant 0$, for all $n \geqslant 0$. Show that $(X_n)_{n \geqslant 0}$ converges almost surely to a finite limit on the set $\{A_\infty < \infty\}$.

c) We no longer suppose $X_n \geqslant 0$. Rather, assume there exists a positive constant c such that $|X_{n+1} - X_n| \leqslant c$, for all $n \geqslant 0$. Prove the almost-sure inclusion:

$$\Big\{\sup_{n \geqslant 0} X_n < \infty\Big\} \subset \Big(\big\{(X_n)_{n \geqslant 0} \text{ converges in } \mathbb{R}\big\} \cap \{A_\infty < \infty\}\Big).$$

3

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, write $\mathbb{E}_{\mathbb{P}}$ for the expectation operator associated with $\mathbb{P}$. Let $\mathbb{Q}$ be a probability on $(\Omega, \mathcal{F})$ absolutely continuous with respect to $\mathbb{P}$, with Radon-Nikodym derivative $D \in \mathbb{L}^1(\mathbb{P})$; its associated expectation operator $\mathbb{E}_{\mathbb{Q}}$ is given by

$$\mathbb{E}_{\mathbb{Q}}[f] = \mathbb{E}_{\mathbb{P}}[D\, f]$$

for any bounded or non-negative random variable $f : \Omega \to \mathbb{R}$. Let $(X_n)_{n\geqslant 1}$ be a sequence of random variables independent and identically distributed under $\mathbb{P}$, with a second moment. Write $m = \mathbb{E}_{\mathbb{P}}[X_1]$ and $\sigma^2 = \mathbb{E}_{\mathbb{P}}\big[(X_1 - m)^2\big]$. We want to prove that the central limit theorem also holds under $\mathbb{Q}$:

$$\overline{S}_n := \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} (X_i - m) \xrightarrow[\text{under } \mathbb{Q}]{\text{law}} \mathcal{N}(0,1). \tag{1}$$

a) Show by a simple example that two random variables can be independent under a probability $\mathbb{P}$ and non-independent under some probability measure $\mathbb{Q}$ absolutely continuous with respect to $\mathbb{P}$.

b) Introducing the filtration $\mathcal{G}_k = \sigma(X_1, \ldots, X_k), k \geqslant 1$, and the random variables $D_k = \mathbb{E}[D|\mathcal{G}_k]$, prove that $\mathbb{E}_{\mathbb{Q}}\big[f(\overline{S}_n)\big]$ converges to $\mathbb{E}_{\mathbb{Q}}\big[f(N)\big]$ as n goes to infinity, where N is under $\mathbb{Q}$ a centred normal random variable with unit variance and $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded. Conclude the proof of the convergence result (1).

4

Let $(X_t)_{t\geqslant 0}$ be a Lévy process starting from 0. Recall $X_{t-} = \lim_{s\uparrow t} X_s$, and set, for $t > 0$, $\Delta X_t = X_t - X_{t-}$. *We say that X has bounded jumps if we have almost surely $\sup_{t>0} |\Delta X_t| \leqslant C$, for some finite constant C.* Also, given any positive constant c, define by induction the stopping times

$$T_1 = \inf\{t \geqslant 0\,;\, |X_t| \geqslant c\} \quad \text{and} \quad T_{n+1} = \inf\{t \geqslant T_n\,;\, |X_t - X_{T_n}| \geqslant c\}, \text{ for } n \geqslant 1.$$

a) Fix any $c > 0$. Prove that there exists a constant $\alpha \in [0,1)$ such that $\mathbb{E}\big[e^{-T_n}\big] = \alpha^n$, for all $n \geqslant 1$.

b) Suppose X has bounded jumps, with jump bound c. By estimating $\mathbb{P}\big(|X_t| \geqslant 2cn\big)$, or by any other method, prove that all the moments of $|X_t|$ are finite.

5

Given a Brownian motion $(B_s)_{s\geqslant 0}$, write $S_t = \sup_{0\leqslant s\leqslant t} B_s$, for each $t > 0$. All stopping times and martingales are considered with respect to the filtration $(\mathcal{F}_s)_{s\geqslant 0}$ generated by $(B_s)_{s\geqslant 0}$. Fix $\varepsilon > 0$ and define by induction the stopping times $T_0(\varepsilon) = T'_1(\varepsilon) = 0$,

$$T_n(\varepsilon) = \inf\{s \geqslant T'_n(\varepsilon)\,;\, S_s - B_s > \varepsilon\} \quad \text{and}$$

$$T'_{n+1}(\varepsilon) = \inf\{s \geqslant T_n(\varepsilon)\,;\, S_s - B_s = 0\}, \quad \text{for } n \geqslant 1.$$

The times $T_n(\varepsilon)$ are the successive times at which $S - B$ achieves an upcrossing of $[0, \varepsilon)$. Define

$$U_t(\varepsilon) = \sup\{k \geqslant 0\,;\, T_k(\varepsilon) \leqslant t\},$$

the number of upcrossing made by $S - B$ before t. Draw a picture of all the situation.

a) (i) Set $H_a = \inf\{s \geqslant 0 : B_s = a\}$. Given two positive constants a and b, prove that

$$\mathbb{P}\big(S_{T_1(\varepsilon)} > a+b \,\big|\, S_{T_1(\varepsilon)} > a\big) = \mathbb{P}\big(S_u - B_u \leqslant \varepsilon \text{ for } H_a \leqslant u \leqslant H_{a+b} \,\big|\, S_u - B_u \leqslant \varepsilon \text{ for } u \leqslant H_a\big).$$

(ii) Prove that $S_{T_1(\varepsilon)}$ has an exponential law with mean ε.

b) (i) Prove that the random variables $S_{T_1(\varepsilon)}, S_{T_2(\varepsilon)} - S_{T_1(\varepsilon)}, S_{T_3(\varepsilon)} - S_{T_2(\varepsilon)}, \ldots$ are independent identically distributed exponential random variables with mean ε.

(ii) For $a > 0$, set $H_a = \inf\{s \geqslant 0\,;\, B_s = a\}$. Deduce from **(i)** that

$$U_{H_a}(\varepsilon) = \sup\{k \geqslant 0\,;\, S_{T_k(\varepsilon)} \leqslant a\}$$

has a Poisson distribution with mean $\frac{a}{\varepsilon}$.

6

Let B be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $t_0^n < \cdots < t_n^n$ be for each $n \geqslant 1$ a finite sequence of points of $[0,1]$, with $t_0^n = 0$ and $t_n^n = 1$. Set $h_n = \max\{t_i^n - t_{i-1}^n\,;\, i = 1, \ldots, n\}$, and define

$$[B]^n := \sum_{i=0}^{n-1} \big(B_{t_{i+1}^n} - B_{t_i^n}\big)^2.$$

We set $\mathcal{F}_n = \sigma\big([B]^n, [B]^{n+1}, \ldots\big)$, for $n \geqslant 1$. Suppose h_n decreases to 0.

a) Prove that the random variables $[B]^n$ converge in $\mathbb{L}^2$ to the constant 1.

b) Deduce from **a)** that if $\sum_{n\geqslant 1} h_n < \infty$, then $[B]^n$ converges almost surely to 1.

c) Suppose the sequence $\big\{t_0^{n+1}, \ldots, t_{n+1}^{n+1}\big\}$ is obtained from the sequence $\big\{t_0^n, \ldots, t_n^n\big\}$ by adding a new point, say t_i^{n+1}. Suppose we have proved that

$$\mathbb{E}\big[[B]^n \big| \mathcal{F}_{n+1}\big] = [B]^{n+1}, \tag{1}$$

for all $n \geqslant 0$. Prove that $[B]^n$ converges almost surely to the constant 1.

d) Prove (1). You may proceed by the following intermediate steps, which should be proved.

(i) Show that there exists a Brownian motion B' and a Bernoulli random variable ν such that B' and ν are independent, $\mathbb{P}(\nu = \pm 1) = \frac{1}{2}$, and

$$B_s = B'_{\min(s, t_i^{n+1})} + \nu\big(B'_s - B'_{\min(s, t_i^{n+1})}\big)\,.$$

(ii) Show that $[B']^k = [B]^k$, for all $k \geqslant n+1$, and compute $[B]^n - [B]^{n+1}$ in terms of $[B']^n - [B']^{n+1}$ and ν.

END OF PAPER