UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 10 June, 2011 9:00 am to 12:00 pm

PAPER 27

STOCHASTIC NETWORKS

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider an $M/M/\infty$ queue with servers numbered $1, 2, \dots$. On arrival a customer chooses the lowest numbered server which is free. Calculate the equilibrium probability that j out of the first n servers are busy. For what fraction of time is server k busy?

[You may find it helpful to express your answer in terms of Erlang's formula.]

Car parking spaces at a park and ride site are arranged as a circular sector, and are labelled (n, m), $n = 1, 2, \dots$; $m = 1, 2, \dots n$, where n indicates the distance to walk to the bus stop. An arriving car parks in one of the free spaces closest to the bus stop.

Cars arrive at a Poisson process of rate ν, and parking times are exponentially distributed with unit mean and are independent of each other and of the arrival process. Find the stationary probability distribution for the distance a car has to park from the bus stop.

Suppose now that to enter the car park cars must pass through a set of ticket barriers, which acts as an $M/M/s$ queue with service times independent of the arrival process and of parking times. Does this affect the stationary distribution for the distance a car has to park from the bus stop? Justify your answer.

2

Customers of class i arrive at a queuing network in independent Poisson streams of rate ρ_i, for $i = 1, 2, 3$. Customers of class i join queue i, and leave the network after service at queue i, for $i = 1, 2$. Customers of class 3 join queue 1 when they arrive at the network, and join queue 2 upon leaving queue 1. All service times are independent and exponentially distributed with unit mean, and are independent of the external arrival process. The service discipline at each queue is first come first served.

Show that in equilibrium the streams of customers of classes 1 and 3 leaving queue 1 form independent Poisson processes. Deduce the stationary distribution of the number of customers in queue 2.

Prove that in equilibrium the probability that the network contains n_i customers of class i, $i = 1, 2, 3$, is

$$B \binom{n_1 + n_2 + n_3 + 1}{n_3} \rho_1^{n_1} \rho_2^{n_2} \rho_3^{n_3} \quad n_1, n_2, n_3 \geqslant 0$$

provided a stability condition, to be stated, is satisfied, and determine the normalizing constant B.

3

Write an essay on mathematical models of loss networks. Your essay should cover the following topics, but need not be restricted to them:

(i) the stationary distribution for a loss network operating under fixed routing;

(ii) a limiting regime where arrival rates and capacities increase together, for a fixed network topology.

4

Define a *Wardrop equilibrium* for the flows in a congested network.

Show that if the delay $D_j(y_j)$ at link j is a continuous, strictly increasing function of the throughput, y_j, of link j then a Wardrop equilibrium exists and solves an optimization problem of the form

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{j \in J} \int_0^{y_j} D_j(u)du \\ \text{over} & x \geqslant 0,\ y \\ \text{subject to} & Hx = f, \quad Ax = y\ , \end{array}$$

where $f = (f_s, s \in S)$ and f_s is the (fixed) aggregate flow between source-sink pair s. What is the interpretation of the matrices A and H? Are the equilibrium throughputs, y_j, unique? Are the equilibrium flows, x_s unique?

Suppose now that the aggregate flow between source-sink pair s is not fixed, but is a continuous, strictly decreasing function $B_s(\lambda_s)$, where λ_s is the minimal delay over all routes serving the source-sink pair s, for each $s \in S$. For the extended model, show that an equilibrium exists and solves the optimization problem

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{j \in J} \int_0^{y_j} D_j(u)du - G(f) \\ \text{over} & x \geqslant 0,\ y, f \\ \text{subject to} & Hx = f, \quad Ax = y\ , \end{array}$$

for a suitable choice of the function $G(f)$, to be determined. Are the equilibrium source-sink flows, f_s, unique?

5

Let J be a set of resources, and R a set of routes, where a route $r \in R$ identifies a subset of J. Let C_j be the capacity of resource j, and suppose the number of flows in progress on each route is given by the vector $n = (n_r, r \in R)$. Define a proportionally fair rate allocation in terms of the solution to an optimization problem, to be carefully stated.

Consider a network with resources $J = \{1, 2, 3, 4\}$, each of unit capacity, and routes $R = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$. Given $n = (n_r, r \in R)$, find the rate x_r of each flow on route r, for each $r \in R$, under a proportionally fair rate allocation. Show, in particular, that if $n_{\{1,2\}} > 0$ then

$$x_{\{1,2\}} n_{\{1,2\}} = \frac{n_{\{1,2\}} + n_{\{3,4\}}}{n_{\{1,2\}} + n_{\{2,3\}} + n_{\{3,4\}} + n_{\{4,1\}}} .$$

Suppose now that flows describe the transfer of documents through a network, that new flows originate as independent Poisson processes of rates ρ_r, $r \in R$, and that document sizes are independent and exponentially distributed with unit mean for each route $r \in R$. Determine the transition rates of the resulting Markov process $n = (n_r, r \in R)$. Show that the stationary distribution of the Markov process $n = (n_r, r \in R)$ takes the form

$$\pi(n) = B^{-1} \binom{n_{\{1,2\}} + n_{\{2,3\}} + n_{\{3,4\}} + n_{\{4,1\}}}{n_{\{1,2\}} + n_{\{3,4\}}} \prod_{r \in R} \rho_r^{n_r} ,$$

provided it exists.

END OF PAPER