UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2011 9:00 am to 12:00 pm

PAPER 26

MODULAR FORMS

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let Λ be a lattice in $\mathbb{C}$. Define the Weierstraß $\wp$-function associated to Λ, and show that it is an elliptic function with respect to Λ. [*You may assume without proof the convergence of the series* $\sum' |\omega|^{-\sigma}$ *for* $\sigma > 2$.]

Compute the Laurent series of $\wp(z)$ at the origin in terms of the constants

$$G_k(\Lambda) = \sum_{0 \neq \omega \in \Lambda} \frac{1}{\omega^k}$$

and deduce that $\wp(z)$ satisfies the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

where $g_2 = 60G_4(\Lambda)$, $g_3 = 140G_6(\Lambda)$.

Suppose that $G_6(\Lambda) = 0$. Show that for some $\omega \in \Lambda \setminus 2\Lambda$,

$$\frac{1}{\wp(z)} = \frac{2\wp(z - \omega/2)}{\wp''(\omega/2)}.$$

Give an example of a lattice Λ for which $G_6(\Lambda) = 0$, and find the corresponding ω.

2

Assuming the dimension formula for the space $M_k(\Gamma(1))$, show that every modular form of level one with integral Fourier coefficients may be expressed as a polynomial, with integer coefficients, in E_4, E_6 and $\Delta = 12^{-3}(E_4^3 - E_6^2)$.

Deduce that there is a unique basis $\{g_0, \ldots, g_{d-1}\}$ for $M_k(\Gamma(1))$ whose Fourier coefficients satisfy

$$a_n(g_j) = \begin{cases} 1 & \text{if } j = n \\ 0 & \text{if } j \neq n \text{ and } n < d \end{cases}$$

where $d = \dim M_k(\Gamma(1))$. Show that $a_n(g_j) \in \mathbb{Z}$ for all j, n.

Let T_n be the n-th Hecke operator acting on $S_k(\Gamma(1))$. Show that for every $n \geqslant 1$,

$$T_n = \sum_{j=1}^{d-1} a_n(g_j) T_j.$$

[*Hint: The q-expansions of* E_4 *and* E_6 *are* $1 + 240\sum \sigma_3(n)q^n$ *and* $1 - 504\sum \sigma_5(n)q^n$.]

3

(i) Let $f = \sum_{n\geqslant 1} a_n q^n \in S_k(\Gamma_0(N))$. Show that $f(z) = O(y^{-k/2})$ on $\mathfrak{H}$, and that the Mellin transform formula

$$(2\pi)^{-s}\Gamma(s)\sum_{n\geqslant 1} a_n n^{-s} = \int_0^\infty f(iy)y^{s-1}\,dy$$

holds whenever $\mathrm{Re}(s) > k/2$. Deduce that

$$\Lambda(f,s) = N^{s/2}(2\pi)^{-s}\Gamma(s)\sum_{n\geqslant 1} a_n n^{-s}$$

has an analytic continuation to all of $\mathbb{C}$ satisfying the functional equation

$$\Lambda(f,k-s) = (-1)^{k/2}\Lambda(f|_k W_N, s).$$

(ii) Let $f \in S_k(\Gamma(1))$, and let $\zeta = e^{2\pi i p/N}$ be a primitive N-th root of unity. By considering $g(z) = f(z + p/N)$ show that the function

$$\Lambda(f,\zeta,s) = N^s(2\pi)^{-s}\Gamma(s)\sum_{n\geqslant 1}\zeta^n a_n n^{-s}$$

satisfies the functional equation

$$\Lambda(f,\zeta,s) = (-1)^{k/2}\Lambda(f,\zeta',k-s)$$

with $\zeta' = e^{2\pi i q/N}$, $pq \equiv -1 \pmod N$.

4

Let $\Gamma \subset SL_2(\mathbb{Z})$ be a subgroup of finite index containing -1.

(i) Obtain the formula

$$g = 1 + \frac{n}{12} - \frac{\nu_\infty}{2} - \frac{\nu_2}{4} - \frac{\nu_3}{3}$$

for the genus of the Riemann surface $\widehat{\Gamma\backslash\mathfrak{H}}$, where ν_∞ is the number of cusps of Γ, and ν_r ($r = 1,\,2$) is the number of Γ-equivalence classes in $\mathfrak{H}$ of points whose stabiliser in $\overline{\Gamma}$ has order r.

(ii) Show that if $\Gamma = \Gamma(N)$ for $N > 1$ then $\nu_2 = \nu_3 = 0$. Deduce that $g(\widehat{\Gamma(N)\backslash\mathfrak{H}}) = 0$ if $N \leqslant 5$.

(iii) Explain what is meant by a *fundamental domain* for Γ. Write down a fundamental domain for $\Gamma(2)$. By considering subgroups of index 2 in $\Gamma(2)$, or otherwise, show that there exists a pair (Γ_1, Γ_2) of distinct subgroups of $\Gamma(1)$ (each containing -1) having a common fundamental domain.

5

Write an essay on EITHER

(i) the theory of Hecke operators for modular forms of level one; OR

(ii) Atkin-Lehner theory for $\Gamma_0(N)$.

END OF PAPER