UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS   Part III

Tuesday, 7 June, 2011   9:00 am to 11:00 am

PAPER 24

LOCAL FIELDS

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let k be a finite field. Define the field of Laurent power series $K = k((t))$. Show that there is a non-archimedean absolute value $|\cdot|$ on K such that
(i) $R = k[[t]]$ is the completion of $k[t]$,
(ii) $K = k((t))$ is the completion of $k(t)$,
(iii) $\#(R/fR) = |f|^{-1}$ for all $0 \neq f \in R$.

(b) Let K be a field complete with respect to a discrete valuation. Show that K is locally compact if and only if it has finite residue field.

2

Let K be a finite Galois extension of $\mathbb{Q}_p$ of degree n.
(a) Show that if non-trivial absolute values $|\cdot|_1$ and $|\cdot|_2$ on K induce the same topology then there exists $c > 0$ such that $|x|_1 = |x|_2^c$ for all $x \in K$.
(b) Show that if $|\cdot|$ is an absolute value on K extending the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}_p$ then

$$|x| = |N_{K/\mathbb{Q}_p}(x)|_p^{1/n}$$

for all $x \in K$. [*You should prove any results you need about equivalence of norms on vector spaces.*]

3

(a) State and prove a version of Hensel's Lemma.

(b) Determine the number of solutions of $x^3 - 11x + 40 = 0$ in $\mathbb{Z}_p$ for $p = 2, 3, 5$.

(c) Let $L \supset K$ be finite extensions of $\mathbb{Q}_p$. Show that if L/K is unramified then L/K is Galois.

4

Let K be a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}_K$ and residue field k of order q. Assuming any properties you need of the Teichmüller map, prove that
(a) $\mathcal{O}_K^*$ contains a subgroup of finite index isomorphic to $(\mathcal{O}_K, +)$.
(b) If $e(K/\mathbb{Q}_p) < p - 1$ then K contains exactly $q - 1$ roots of unity.
(c) $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^3 \cong \mathbb{Z}/3\mathbb{Z}$ or $(\mathbb{Z}/3\mathbb{Z})^2$.

5

(a) Show that if $m \in \mathbb{Z}$ with $m \equiv 2 \pmod 9$ then $\mathbb{Q}_3(\zeta_3, \sqrt[3]{m})/\mathbb{Q}_3$ is a totally ramified extension of degree 6.

(b) Write an essay on higher ramification groups and illustrate by computing them for the example in part (a).

END OF PAPER