UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 9 June, 2011 9:00 am to 12:00 pm

PAPER 21

TOPOS THEORY

*Attempt no more than **THREE** questions.*

*There are **SEVEN** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Let G be a group. Prove that the category $\mathbf{B}G = [G^{\text{op}}, \mathbf{Set}]$ of right G-actions and equivariant maps between them is an elementary topos, by explicitly constructing its finite limits, exponentials and subobject classifier.

(ii) Let G be a topological group and $\mathbf{B}G$ the category of continuous G-sets (i.e. the category having as objects the right G-actions $X \times G \to X$ which are continuous when X is endowed with the discrete topology and $X \times G$ is endowed with the product topology, and as arrows the equivariant maps between them). Show that $\mathbf{B}G$ is an elementary topos, by constructing a right adjoint $Z_G : \mathbf{B}G^d \to \mathbf{B}G$ to the inclusion functor $\mathbf{B}G \to \mathbf{B}G^d$, where G^d is the group G endowed with the discrete topology [*Hint:* Define Z_G on the objects by sending an object $(X, r : X \times G^d \to X)$ of $\mathbf{B}G^d$ to the restricted action $(X_r, r| : X_r \times G \to X_r)$ where $X_r = \{x \in X \mid \{g \in G^d \mid r(x, g) = x\}$ is open in $G\}$].

2

Explain what is meant by *local operator* on a topos, and sketch the proof that the following three concepts are equivalent for any topos $\mathcal{E}$:

(i) local operators on $\mathcal{E}$;

(ii) universal (i.e. pullback-stable) closure operations on subobjects in $\mathcal{E}$;

(iii) full reflective subcategories of $\mathcal{E}$ with cartesian (i.e. finite-limit-preserving) reflector.

3

Show that for any site $(\mathcal{C}, J)$ and any Grothendieck topos $\mathcal{E}$, there is an equivalence between the category of geometric morphisms $\mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)$ and the category of J-continuous flat functors $\mathcal{C} \to \mathcal{E}$. Show further that the geometric morphism corresponding to a flat J-continuous functor $F : \mathcal{C} \to \mathcal{E}$ is surjective if and only if F is *cover-reflecting*, in the sense that, for any sieve S in $\mathcal{C}$ on an object c of $\mathcal{C}$, F sends S to an epimorphic family if and only if S is J-covering [*You may use the surjection-inclusion factorization of a geometric morphism, if you want*].

4

Let σ be the sequent

$$(\top \vdash_{x,y} (x \Rightarrow y) \vee (y \Rightarrow x)).$$

written in the theory of Heyting algebras.

(i) Show that if σ is valid in the internal Heyting algebra of a topos $\mathcal{E}$ given by its subobject classifier $\Omega_{\mathcal{E}}$ then $\mathcal{E}$ satisfies De Morgan's law, i.e. the sequent

$$(\top \vdash_{x} \neg x \vee \neg\neg x).$$

is valid in $\Omega_{\mathcal{E}}$.

(ii) Show that σ is valid in the internal Heyting algebra $\Omega_{\mathbf{Sh}(L)}$ of a topos $\mathbf{Sh}(L)$ of sheaves on a locale L given by its subobject classifier if and only if σ is valid in L, regarded as a model of the theory of Heyting algebras in **Set**, equivalently if for any elements $a, b \in L$, $1_L = (a \Rightarrow b) \vee (b \Rightarrow a)$ in L (where 1_L is the top element of L and $\Rightarrow$ denotes the Heyting implication in L).

(iii) Give necessary and sufficient conditions on a small category $\mathcal{C}$ for the sequent σ to be valid in the algebra $\Omega_{[\mathcal{C}^{\text{op}},\mathbf{Set}]}$ of the presheaf topos $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$.

5

Let $\mathbb{T}$ be a geometric theory over a signature Σ.

(i) Show that, for any small category $\mathcal{C}$, we have an equivalence of categories

$$\mathbb{T}\text{-mod}([\mathcal{C}, \mathbf{Set}]) \simeq [\mathcal{C}, \mathbb{T}\text{-mod}(\mathbf{Set})].$$

(ii) Show that, for any topological space X, a Σ-structure M in $\mathbf{Sh}(X)$ is a $\mathbb{T}$-model if and only if each $x^*(M)$, $x \in X$, is a $\mathbb{T}$-model in **Set**, where $x^* : \mathbf{Sh}(X) \to \mathbf{Set}$ is the stalk functor associated with x (i.e. the inverse image functor of the geometric morphism $\mathbf{Set} \to \mathbf{Sh}(X)$ corresponding to the point x of X).

(iii) Show that if $\mathbb{T}$ is a Horn theory over Σ then for any topological space X, a Σ-structure M in $\mathbf{Sh}(X)$ is a $\mathbb{T}$-model if and only if for every open set U of X, $M(U)$ is a $\mathbb{T}$-model in **Set**; is this true also for an arbitrary coherent theory $\mathbb{T}$?

6

Define the notions of geometric theory and classifying topos, and sketch the proof that every geometric theory has a classifying topos.

(i) Can a geometric theory have two inequivalent classifying toposes?

(ii) Can two distinct geometric theories have equivalent classifying toposes?

7

Write an essay on the sense in which Grothendieck toposes can serve as 'bridges' for unifying Mathematics. [*Detailed proofs are not required, but you should illustrate your arguments with appropriate examples.*]

END OF PAPER