UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 8 June, 2011 9:00 am to 12:00 pm

PAPER 20

CATEGORY THEORY

*Attempt **ONE** question from Section I, and **TWO** from Section II.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1

Define the notion of *monad*, and explain how every adjunction gives rise to a monad. Define the *Kleisli category* associated with a given monad $\mathbb{T}$; prove that it is a category, and that it comes equipped with an adjunction inducing $\mathbb{T}$. Sketch the proof that the Kleisli category is initial in the category of adjunctions inducing $\mathbb{T}$.

2

Peter Freyd once suggested that the purpose of category theory is 'to show that which is trivial is trivially trivial'. Write an essay arguing *either* for *or* against this assertion; it is suggested that such an essay should include some reference to the Yoneda Lemma and/or the Adjoint Functor Theorems.

SECTION II

3

Define a *balanced* category. If $F : \mathcal{C} \to \mathcal{D}$ is a faithful functor and $\mathcal{C}$ is balanced, show that F reflects isomorphisms.

Let $(F : \mathcal{C} \to \mathcal{D} \dashv G : \mathcal{D} \to \mathcal{C})$ be an adjunction with unit η and counit ϵ. Show that F is faithful if and only if η is a (pointwise) monomorphism. If both η and ϵ are monomorphisms and $\mathcal{C}$ is balanced, show that F is full and faithful. Give an example to show that the hypothesis on $\mathcal{C}$ cannot be omitted.

4

Let $\mathcal{C}$ be a category, and let $\mathcal{D}$ be a full subcategory of the category $[\mathcal{C}, \mathcal{C}]$ of endofunctors of $\mathcal{C}$ which is closed under composition and contains the identity functor. Suppose $\mathcal{D}$ has a terminal object T; show that T carries a unique monad structure $\mathbb{T}$, and that if $\mathbb{S}$ is any other monad whose functor part lies in $\mathcal{D}$, there is a forgetful functor $\mathcal{C}^{\mathbb{T}} \to \mathcal{C}^{\mathbb{S}}$.

Now let $\mathcal{C} = \mathbf{Set}$ and let $\mathcal{D}$ be the category of functors $\mathbf{Set} \to \mathbf{Set}$ which preserve finite coproducts. Show that $\mathcal{D}$ has a terminal object, namely the functor β which assigns to a set A the set of all ultrafilters on A (that is, maximal proper filters in the lattice PA of all subsets of A).

5

Let $\mathcal{C}$ be a small category and $F : \mathcal{C} \to \mathbf{Set}$ a functor. Show that F may be represented as the colimit in $[\mathcal{C}, \mathbf{Set}]$ of a diagram of shape $(1 \downarrow F)^{\mathrm{op}}$ whose vertices are representable functors. [Here 1 denotes a singleton set $\{*\}$.]

Now suppose $\mathcal{C}$ has finite limits. Show that the following conditions are equivalent:

(i) F preserves finite limits.

(ii) For any set A, the category $(A \downarrow F)$ has finite limits.

(iii) $(1 \downarrow F)^{\mathrm{op}}$ is filtered.

(iv) F is expressible as a (small) filtered colimit of representable functors.

[You may assume the result that filtered colimits commute with finite limits in $\mathbf{Set}$.]

6

Define the notions of semi-additive and additive category, and show that finite products and coproducts coincide in a semi-additive category.

Recall that, in any category, a parallel pair $f, g\colon A \rightrightarrows B$ is said to be *reflexive* if there exists $r\colon B \to A$ such that $fr = gr = 1_B$. Show that any reflexive pair (f, g) in an additive category $\mathcal{C}$ has the structure of an *internal groupoid*: that is, for each object C, the set $\mathcal{C}\,(C, B)$ is the set of objects of a groupoid whose morphisms are the members of $\mathcal{C}\,(C, A)$, with 'domain' and 'codomain' given by composition with f and g respectively.

By considering a suitable parallel pair in the category of commutative monoids, or otherwise, show that this result does not hold in all semi-additive categories.

END OF PAPER