UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2011 1:30 pm to 4:30 pm

PAPER 2

DECISION PROBLEMS IN GROUP THEORY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let G and H be finitely presented groups. Prove or give a counter example to each of the following, stating clearly any theorems you use.

(i) If G and H have soluble word problem, then so does $G * H$.

(ii) The free group on 5 generators can be embedded in the free group on 2 generators.

(iii) If G has soluble word problem, then so does any finitely generated homomorphic image.

(iv) If $f, g, \in G$ have order 4, then they are conjugate in some HNN-extension of G.

(v) Every countable group can be embedded in a finitely presented group.

(vi) If G has soluble word problem, then so does any HNN-extension of G.

2

State the Messuage Lemma and use it to prove that there is no algorithm to determine whether or not an arbitrary finitely presented group is non-abelian. You may assume the existence of a non-abelian finitely presented group with insoluble word problem.

3

Let $G = \langle x, y : [x, y^{\pm 1}, x] = [x^{-1}, y^{\pm 1}, x] = [x, y^{\pm 1}, y] = [x^{-1}, y^{\pm 1}, y] = 1 \rangle$. Describe $G/\gamma_2(G)$ and $\gamma_2(G)/\gamma_3(G)$, and prove that G is nilpotent class 2. Illustrate how to solve the word problem for G, giving examples.

You may assume that $[b, a] = [a, b]^{-1}$, $[ab, c] = [a, c]^b [b, c]$, and $[a, bc] = [a, c][a, b]^c$.

4

(i) Let A and B be isomorphic subgroups of a finite group G. Prove that A and B are conjugate in some finite group H containing G.

(ii) Let $g, x_1, \ldots, x_n \in G$ and N be the normal subgroup of G generated by g. If $N \cap \langle x_1, \ldots, x_n \rangle = \{1\}$ and $\langle x_1, \ldots, x_n \rangle$ is free of rank n, prove that $\langle x_1^{-1} g, x_2^{-1} g^2, \ldots, x_n^{-1} g^n \rangle$ is isomorphic to $\langle g^{-1} x_1, \ldots, g^{-1} x_n \rangle$.

5

Give an outline of the main ideas involved in any construction (and proof) of a finitely presented group with insoluble word problem.

6

(i) Let $G = \langle a, b, c : a^2 = 1,\ c^{-1}bc = b^{-1} \rangle$. Construct a finitely presented group H containing G such that b belongs to the normal subgroup of H generated by a, proving your result.

(ii) Let $w := xyzy^{-4}x^{-1}zy^2x^3y$, and $G = \langle x, y, z : w = 1 \rangle$. By using a different set of generators of G, express G as an HNN-extension of a one-relator group K, where the defining word for K has length (in the new generators) less than the length of w (in x, y, z), quoting any theorems you use.

END OF PAPER