UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 9 June, 2011 9:00 am to 12:00 pm

# PAPER 19

# GALOIS COHOMOLOGY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

Let $K$ be a perfect field (so that all finite extensions of $K$ are separable) and let $A$ be a central simple algebra over $K$. Give the definition of the index $\mathrm{ind}_K A$ of $A$.

Show that $\mathrm{ind}_K A$ is the gcd of the degrees $[L : K]$ of the finite field extensions $L/K$ that split $A$. (You may assume the double-centralizer theorem, as well as the formula $[B : K] = [A : C_A(B)]$ as long as these are clearly stated.) This gives an alternate definition of the index. How would you define the index of a Severi-Brauer variety $X$ over $K$?

Recall that the period of $A$, denoted $\mathrm{per}_K(A)$, is the order of its class $[A] \in Br(K)$. Using a Corestriction-Restriction argument, show that $\mathrm{per}_K(A)$ divides $\mathrm{ind}_K A$. Show that $\mathrm{per}_K(A)$ and $\mathrm{ind}_K A$ have the same prime factors, i.e. if $p$ is a prime, then $p$ divides $\mathrm{per}_K(A)$ if and only if $p$ divides $\mathrm{ind}_K A$. [*Hint : you may show that if $p$ does not divide $\mathrm{per}_K(A)$, then $A$ is split by a finite extension $L/K$ of degree prime to $p$.*]

**2**

Let $K$ be a field and let $A$ be a central simple algebra over $K$.

1. Give the definition of the reduced norm $\mathrm{Nrd} : A \to K$ and show that it doesn't depend on choices.

2. Define the $C_r$-property for fields and give examples of $C_1$-fields [*no proofs are required* ].

3. Prove that if $K$ is $C_1$, then $Br(K) = 0$.

4. If $K$ is the function field of a curve over a finite field, show that $A$ is split by a cyclic extension of $K$.

5. Let $K$ be a $C_2$-field and let $D$ be a central division algebra over $K$. Show that $\mathrm{Nrd} : A \to K$ is surjective.

**3**

1. Give the definition of the $p$-cohomological dimension of a field. If $K$ is a field of characteristic $p > 0$, show that $cd_p(K) \leqslant 1$. Any result about cohomological dimension that you may use must be proven.

2. Let $L/K$ be a purely inseparable extension of $K$ (this means that for all $\alpha \in L$, there is an integer $n$ such that $\alpha^{p^n} \in K$). Show that the restriction map $Br(K) \to Br(L)$ is surjective. Show that its kernel is $p$-primary torsion.

**4**

Let $K$ be a field. Fix a separable closure $\bar{K}$. Let $X$ be a variety over $K$ such that $\bar{X} := X \times_{\text{Spec } K} \text{Spec } \bar{K}$ is a smooth projective variety over $\bar{K}$.

Explain briefly why there is an exact sequence

$$0 \to \bar{K}^{\times} \to \bar{K}(X)^{\times} \to \text{Div}\bar{X} \to \text{Pic}\bar{X} \to 0.$$

Use this exact sequence to establish the exact sequence

$$0 \longrightarrow \text{Pic}X \xrightarrow{i} (\text{Pic}\bar{X})^{\text{Gal}(\bar{K}/K)} \xrightarrow{\delta} BrK \xrightarrow{Res} Br(K(X)),$$

where $i$ is the natural map and $Res$ is the restriction map.

Now, suppose that $X$ is a Severi-Brauer variety over $K$. Describe explicitly the map $\delta$ [*no proof is required* ].

If $C$ is a conic over $K$ with $C(K) = \emptyset$, show that $\text{Ker}(Br(K) \to Br(K(C))) = \mathbf{Z}/2\mathbf{Z}$.

## END OF PAPER