UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 10 June, 2011 9:00 am to 12:00 pm

PAPER 18

SPECTRAL GEOMETRY

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Give, in terms of appropriate coordinate systems, three different expressions for the Laplacian acting on smooth functions at a point p on a Riemannian manifold.

Define what it means for a map between Riemannian manifolds to be a Riemannian submersion with totally geodesic fibres.

Identify, with proof, the eigenspaces of the Laplacian for a product $M \times N$ of Riemannian manifolds $(M,g),(N,h)$ with the product metric $g \times h$, in terms of those of (M,g) and (N,h).

2

Show how the subspace $T \leqslant \mathbb{F}_3^4$ spanned by the vectors $(0,1,1,1)$ and $(1,1,-1,0)$ may be used to construct two lattices L^+ and L^- in $\mathbb{R}^4$ such that the flat tori $\mathbb{R}^4/L^{\pm}$ have the same length spectrum but are not isometric.

3

Define the heat kernel, heat trace and heat invariants of a Riemannian manifold. Assuming the existence of the heat kernel for a compact Riemannian manifold, prove its uniqueness.

Deduce that the heat trace determines the spectrum of the Laplacian and vice versa, and that isospectral manifolds have the same dimension.

4

Let T be a finite group of isometries acting freely on the Riemannian manifold (N,g). Express the heat kernel of the orbit space $M = T\backslash N$ in terms of that of N.

Derive Sunada's expression for the heat trace of M and deduce his theorem for producing isospectral pairs of manifolds with a common covering.

Recall that a Riemann surface was defined as a 2-dimensional Riemannian manifold with constant curvature -1. Describe how Sunada's theorem may be used to construct isospectral Riemann surfaces, making explicit any further hypotheses you require and identifying the Euler characteristic of the resulting surfaces.

Suggest how one might obtain isospectral Riemann surfaces of genus 3 and how, if these turn out to be isometric, one might nevertheless produce isospectral non-isometric surfaces of genus 3.

5

Which compact Riemann surfaces with boundary are called X-pieces and Y-pieces? Identify the parameters that determine them up to isometry, including those involved when two Y-pieces form an X-piece.

Describe how a general closed Riemann surface of genus g may be formed from a collection of Y-pieces and identify a set of parameters that suffice to determine the surface up to isometry.

Define Teichmüller space and state Wolpert's Theorem. State the key results required for Buser's proof of Wolpert's Theorem and explain briefly their role in that proof.

END OF PAPER