UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 6 June, 2011 1:30 pm to 4:30 pm

PAPER 17

ALGEBRAIC GEOMETRY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

Algebraic sets are defined over an algebraically closed field k, unless stated otherwise.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $d \geqslant 1$ and $n \geqslant 2$. Let the algebraically closed base field k have characteristic zero. The set of nonzero homogeneous polynomials of degree d in $n+1$ variables modulo scalars is a projective space $\mathbf{P}^N$. What is the number N? Show that the set of irreducible hypersurfaces of degree d in $\mathbf{P}^n$ is an open subset of $\mathbf{P}^N$ in the Zariski topology. Show that the set of smooth irreducible hypersurfaces of degree d in $\mathbf{P}^n$ is a connected open subset of $\mathbf{P}^N$.

2

Let Z be a closed subspace of a topological space X, and let $i : Z \to X$ be the inclusion. For any sheaf E of abelian groups on Z, show that the direct image sheaf i_*E satisfies $H^j(Z, E) \cong H^j(X, i_*E)$ for all $j \geqslant 0$. [*Hint: use the definition of $H^*(Z, E)$. Note that sheaf cohomology is not computed by the Cech complex in this generality.*]

Give an example to show that $H^j(Z, E)$ need not be isomorphic to $H^j(X, i_*E)$ for Z a subspace of X which is not closed.

3

(a) Let X be a smooth projective curve of genus $g \geqslant 1$. You may use that every line bundle L of positive degree d on X has $h^0(X, L) \leqslant d$. Show that every line bundle L of degree $d \geqslant 3$ such that $h^0(X, L)$ is equal to d must be very ample (that is, L gives an embedding of X into some projective space).

(b) Let X be a smooth projective curve of genus $g \geqslant 2$ and degree 4 in some $\mathbf{P}^N$. Show that X is contained in some linear subspace $\mathbf{P}^2 \subset \mathbf{P}^N$. [*Hint: Show that otherwise X could be embedded in $\mathbf{P}^2$ as a plane cubic curve. Part (a) may be helpful.*] As a result, compute the genus of X.

(c) Let X be a smooth curve of degree 4 in $\mathbf{P}^2$. Show that X cannot be embedded in any $\mathbf{P}^N$ as a curve of degree 5. [*Hint: what could N be?*]

4

Define the tensor product $E \otimes F$ of two coherent sheaves on an algebraic set X. Show that $(E \otimes F)(U) \cong E(U) \otimes_{O(U)} F(U)$ for every affine open subset $U \subset X$. [*You may use general theorems on coherent sheaves.*]

Give an example to show that $(E \otimes F)(U)$ need not be isomorphic to $E(U) \otimes_{O(U)} F(U)$ for an open subset U of X that is not affine.

5

Let X be a smooth surface of degree 4 in $\mathbf{P}^3$. Suppose that X contains a smooth elliptic curve C. Show that the normal bundle of C in X is trivial. Show that the line bundle $O(C)$ is basepoint-free on X. Show that there is a morphism $X \to \mathbf{P}^1$ whose fiber over some point in $\mathbf{P}^1$ is C.

END OF PAPER