UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 10 June, 2011 1:30 pm to 4:30 pm

PAPER 16

HODGE THEORY

Attempt no more than ***THREE*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let X be a complex manifold. Show that the *Bott-Chern cohomology* groups

$$H^{p,q}_{BC}(X) = \frac{\{\alpha \in \mathcal{A}^{p,q}(X) | d\alpha = 0\}}{\partial\overline{\partial}\mathcal{A}^{p-1,q-1}(X)}$$

are well defined, and that there are natural morphisms

$$H^{p,q}_{BC}(X) \to H^{p,q}(X) \text{ and } H^{p,q}_{BC}(X) \to H^{p+q}(X, \mathbb{C}).$$

State and prove the $\partial\overline{\partial}$-lemma. Show that if X is compact and Kähler, the map $H^{p,q}_{BC}(X) \to H^{p,q}(X)$ is an isomorphism. Deduce from this that the bidegree decomposition in the Hodge Decomposition Theorem is independent of the choice of Kähler structure.

2

Let X be a projective manifold and $Z \subset X$ be a submanifold. Define and construct the blowup of X along Z $\sigma\colon \widetilde{X}_Z \to X$ and prove that $\widetilde{X}_Z$ is Kähler.

Assuming that the cohomology groups $H^k(X, \mathbb{Z})$ are torsion free, show that the degree k cohomology groups of $\widetilde{X}_Z$ with coefficients in $\mathbb{Z}$ and $\mathbb{C}$ define an Integral Hodge Structure of weight k (you may quote the Hodge Decomposition Theorem).

Describe the Hodge Structure on $(H^k(\widetilde{X}_Z, \mathbb{Z}), H^k(\widetilde{X}_Z, \mathbb{C}))$ in terms of the Hodge structures on the cohomology of X and Z (you may quote without proof any auxiliary result on pullbacks, push-forwards, cohomology of pairs and of fibre bundles).

Compute the Hodge numbers of the blowup of $\mathbb{P}^3$ along the curve $Z = \{x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0\} \subset \mathbb{P}^3$.

3

Show that the Grassmannian $\mathrm{Gr(k,n)}$ of k-planes in $\mathbb{C}^n$ is a complex manifold. Define its tautological vector bundle $\mathcal{S} \to \mathrm{Gr(k,n)}$ and show that $\mathcal{S}$ is a holomorphic vector bundle of rank k.

Let X be a complex manifold and $E \to X$ be a subbundle of rank k of $X \times \mathbb{C}^n$. Show that there is a unique holomorphic map $f: X \to \mathrm{Gr(k,n)}$ such that $E = f^*\mathcal{S}$.

Let $\mathcal{X} \to B$ be a proper holomorphic submersive map between connected complex manifolds. Assume that B is contractible, and that $X = \pi^{-1}(0)$ is a compact Kähler manifold. Define the Hodge bundle $\mathcal{H}^k$, and prove that $\mathcal{H}^k$ is a trivial vector bundle of rank $b_k(X) = \dim H^k(X, \mathbb{C})$ over B.

Assume that $\mathcal{X}_b = \pi^{-1}(b)$ is a compact Kähler manifold for all $b \in B$. Define the Hodge filtration $\{F^pH^k(\mathcal{X}_b, \mathbb{C})\}_{0 \leqslant p \leqslant k}$ on the Betti cohomology $H^k(\mathcal{X}_b, \mathbb{C})$. Let $b^{p,k}(b) = \dim F^pH^k(\mathcal{X}_b, \mathbb{C})$.

Define the period map $\mathcal{P}: B \to \mathrm{Gr(b^{p,k}, H^k(X, \mathbb{C}))}$, and show there is a filtration of $\mathcal{H}^k$ by holomorphic subbundles $F^p\mathcal{H}^k \subset \mathcal{H}^k$, whose fibres over $b \in B$ are $(F^p\mathcal{H}^k)_b = F^pH^k(\mathcal{X}_b, \mathbb{C})$. (You may quote any result from lectures on the behaviour of the functions $b \mapsto b^{p,k}(b)$ or on the period map.)

4

Let $z_1, \cdots, z_n$ be coordinates on $\mathbb{C}^n$, $\Lambda = \mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_{2n}$ a full lattice and X the complex torus $\mathbb{C}^n/\Lambda$. Define the transition matrices $\widetilde{\Omega}$ and $\widetilde{\Pi}$ between the integral and complex structures on X as follows. If

$$\Omega = \begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{2n,1} \\ \vdots & & \vdots \\ \lambda_{1,n} & \cdots & \lambda_{2n,n} \end{pmatrix}$$

is the matrix whose column vectors are the coordinates of the vectors λ_i for $1 \leqslant i \leqslant 2n$, set $\widetilde{\Omega} = \begin{pmatrix} \Omega \\ \overline{\Omega} \end{pmatrix}$ and $\widetilde{\Pi} = \begin{pmatrix} \Pi & \overline{\Pi} \end{pmatrix}$, with $\widetilde{\Omega}\widetilde{\Pi} = Id$.

Let H be a Hermitian metric with constant coefficients on $\mathbb{C}^n$, and equip X with the induced metric. Describe harmonic forms and the Betti cohomology groups of X in terms of the holomorphic coordinates of $\mathbb{C}^n$.

Let $G = (g_{i,j})_{1 \leqslant i,j \leqslant 2n}$ be a skew-symmetric and integral matrix, and suppose that $\gamma = \Sigma g_{i,j} dx_i \wedge dx_j$ is a real positive definite $(1,1)$-form. Use Π to express γ in terms of the coordinates z_k and show that Ω satisfies the *Riemann bilinear relations*:

$$\begin{cases} i\overline{\Omega}G^{-1\,t}\Omega > 0 \\ \Omega G^{-1\,t}\Omega = 0 \end{cases} \quad (G \text{ integral and skew-symmetric})$$

Show that after a suitable change of coordinates, the matrix of G is:

$$G = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \text{ with } D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}, d_i \in \mathbb{Z}.$$

Show that, if the first n vectors of Λ are identified with the standard basis of $\mathbb{C}^n$, $\Omega = \begin{pmatrix} Id & Z' \end{pmatrix}$, and setting $Z = Z'D$, the Riemann relations become:

$$^tZ = Z, \quad \Im(Z) > 0.$$

Give a criterion for a complex torus to be abelian in terms of Ω. Show that if $n > 1$ projective complex tori depend on $\frac{n(n+1)}{2}$ continuous parameters, so that there are many non abelian tori.

Let C be a compact connected complex curve, show that C is projective. Define the period map $\mathcal{P}$ associated to the weight 1 Hodge structure on $(H^1(C,\mathbb{Z}), H^1(C,\mathbb{C}))$. Define $g = b_1(C)/2$. Show that the polarised period domain of X is contained in the Siegel upper half space $\mathfrak{h}_g$.

5

Let X be a compact Kähler manifold. Show that the map $H^k(X,\mathbb{C}) \to H^k(X,\mathcal{O}_X)$ induced by the inclusion $\mathbb{C} \subset \mathcal{O}_X$ and the projection map $H^k(X,\mathbb{C}) \to H^k(X,\mathcal{O}_X)$ induced by the Hodge decomposition coincide.

Show that the image of the map $\text{Pic}X \to H^2(X,\mathbb{C})$ induced by the exponential sequence is contained in

$$H^{1,1}(X,\mathbb{Z}) = \text{im}(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})) \cap H^{1,1}(X).$$

Use the Hodge decomposition to give a proof of the Lefschetz decomposition on $(1,1)$-classes: prove that $\text{Pic}X \to H^{1,1}(X,\mathbb{Z})$ is surjective.

Show that a compact Kähler manifold X with $h^{2,0}(X) = 0$ is projective. (You may quote without proof Kodaira's Embedding Theorem).

END OF PAPER