UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2011 1:30 pm to 4:30 pm

PAPER 15

MORSE HOMOLOGY

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Define *Morse function.*

(ii) State the *Morse Lemma.*

(iii) Define the *Morse index.*

(iv) Show that being Morse is a C^2-open condition.

(v) State the *parametric transversality theorem.*

(vi) Prove that Morse functions are dense inside the space of continuous functions.

(vii) Prove that if M is a closed manifold with a Morse function which has exactly two critical points, then M is homeomorphic to a sphere.

2

Let M be a closed manifold, and let $f : M \to \mathbb{R}$ be Morse.

(i) Define the *moduli space* of $-\nabla f$ trajectories.

(ii) State the *Transversality Theorem* for these moduli spaces.

From now on, assume that transversality holds for these moduli spaces.

(iii) Prove that the moduli space of $-\nabla f$ trajectories between critical points p, q of index difference $|p| - |q| = 1$ are compact.

(iv) State a theorem describing a natural compactification of the moduli spaces when $|p| - |q| = 2$.

(v) Describe this compactification in the case of $M = \mathbb{R}P^2$. *[You do **not** need to write a formula for f, it suffices that you draw the flowlines for an f that you have chosen]*

3

(i) Define *Fredholm map*, *Fredholm operator*, *Fredholm index.*

(ii) Explain how the operator

$$\partial_s + A_s : W^{1,2}(\mathbb{R}, \mathbb{R}^m) \to L^2(\mathbb{R}, \mathbb{R}^m)$$

arises in Morse homology, where the matrices $A_s \to A_{\pm\infty}$ as $s \to \pm\infty$, and where $A_{\pm\infty}$ is invertible and symmetric.

(iii) Prove that $\partial_s + A_s$ is Fredholm.
[You may quote the Closed Range Lemma *without proof]*

(iv) State a theorem describing the Kernel and Cokernel of $\partial_s + A_s$.

(v) Using (iv), compute the Fredholm index of $\partial_s + A_s$.

4

Let M be a closed manifold.

(i) Define *self-indexing Morse function* and *Morse-Smale metric.*

(ii) State the *Handle Attaching Theorem.*

(iii) Outline the proof of the natural identification

$$MH_*(\text{self-indexing Morse function}) \cong H_*^{\text{cellular}}(M) \qquad (\text{over } \mathbb{Z}/2\mathbb{Z}).$$

[You may state without proof what the cellular differential is in terms of an intersection number, but please describe the generators of the cellular chain complex]

(iv) Define *continuation maps* and state their properties at the chain level and at the homology level.

(v) Deduce from (iv) the *Invariance Theorem* for Morse homology.

(vi) Deduce from (v) a generalization of (iii) for any Morse function.

(vii) Show that generically a function on a closed orientable surface must have at least $2 + 2g$ critical points, where g is the genus of the surface.

5

(i) Define the meaning of *generic subset* (*Baire subset*), and state the *Baire category theorem.*

(ii) State the *Sard-Smale Theorem.*

(iii) State the *Implicit Function Theorem* for maps between Banach manifolds.

(iv) Deduce from (iii) that if a section $F : M \to E$ of a Banach vector bundle is transverse to 0 *[zero section of E]* then $F^{-1}(0) \subset M$ is a submanifold and $T_pF^{-1}(0) = \ker D_pF$.

(v) Suppose that $E \to M \times S$ is a Banach vector bundle, and F is a smooth section, and that for all (m, s) with $F(m, s) = 0$ the following hold:

(1) $D_{(m,s)}F : T_{(m,s)}(M \times S) \to E_{(m,s)}$ is surjective

(2) $D_mF_s : T_mM \to E_{(m,s)}$ is Fredholm of index k *[where $F_s = F(\cdot, s)$]*

(3) $\ker D_{(m,s)}F$ has a closed complement. *[This is a consequence of 1 and 2, but you need not prove it]*

Then prove that $F_s^{-1}(0)$ is a smooth manifold of dimension k for generic $s \in S$. *[You should prove any transversality result you use]*

(vi) Briefly mention the setup in which this result is applied in Morse homology *[when one proves the Transversality Theorem for moduli spaces].*

END OF PAPER