UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 8 June, 2011 9:00 am to 12:00 pm

PAPER 14

DIFFERENTIAL GEOMETRY

Attempt no more than ***THREE*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define the cotangent bundle T^*M for a smooth manifold M. By constructing an appropriate family of charts, show that T^*M has a smooth structure making it into a vector bundle over M. Show further that the vector bundle T^*M can be endowed with an inner product varying smoothly with the fibres.

Now let M be the unit sphere S^2 and consider the subset Y of T^*S^2 consisting of all the unit vectors in cotangent spaces. Show that the vector bundle structure on T^*S^2 induces on Y structure of a principal S^1-bundle over S^2. Is Y isomorphic to a product bundle $S^2 \times S^1$? Justify your answer.

[*Existence of a partition of unity may be assumed without proof, provided the result is accurately stated. You may assume that there are no continuous nowhere vanishing vector fields on* S^2.]

2

Define what is meant by a Lie group G and left-invariant vector fields on G. Show that if X and Y are two left-invariant vector fields on G, then their Lie bracket $[X, Y]$ is left-invariant and that the space of left-invariant vector fields on G is finite dimensional.

Let X_i, $i = 1, \ldots, m$, be a basis of left-invariant vector fields on G. Show that the identities $\omega^i(X_j) = \delta^i_j$ (δ^i_j is the Kronecker delta) determine smooth 1-forms ω^i, $i = 1, \ldots, m$, on G which are linearly independent at each point in G . Show further that the 1-forms ω^i satisfy

$$L_g^*(\omega^i) = \omega^i, \qquad \text{for every } g \in G,$$

where $L_g(h) = gh$ for each $h \in G$. Let C^k_{ij} be a set of real constants determined by $[X_i, X_j] = \sum_k C^k_{ij} X_k$. Prove that, for each k,

$$d\omega^k = - \sum_{1 \leqslant i < j \leqslant m} C^k_{ij} \omega^i \wedge \omega^j.$$

[*You may assume the identity* $d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$, *for a 1-form* ω *and vector fields* X, Y.]

3

Let A be a connection on a vector bundle E. Using local coordinates on the base manifold and a local trivialization of E, give an explicit formula for the covariant derivative d_A induced by A and acting on sections of E. Explain how to extend d_A, using an appropriate version of the Leibniz rule, to the differential forms with values in E and to the differential forms with values in the endomorphism bundle $\operatorname{End} E$. For both cases, include explicit formulae for d_A in local trivializations.

Define the curvature $F(A)$ of a connection A, showing that $F(A)$ is a well-defined 2-form with values in $\operatorname{End} E$. Prove the Bianchi identity $d_A F(A) = 0$

Prove that if E has rank 1 and A is a connection on E and a is a 1-form on the base manifold, then $A + a$ is a connection with curvature $F(A + a) = F(A) + da$. Determine, giving justification, a more general version of the latter formula valid when the rank of E is greater than 1.

4

Define geodesic coordinates on a Riemannian manifold M. Show, stating clearly any preliminary results that you use, that geodesic coordinates exist on a neighbourhood of any point $p \in M$.

State and prove Gauss' Lemma.

[*You may assume without proof that the length of* $|\dot{\gamma}(t)|$ *is constant for any geodesic* $\gamma(t)$.]

5

Let M be an oriented Riemannian manifold with a metric g. Define the volume form ω_g on M, showing that ω_g is well-defined. Define the Hodge star operator $*$ and compute its square on p-forms.

Now suppose that M is compact. Recall that the linear operator δ is defined by $\delta\psi = (-1)^{n(p+1)+1} {*d*}\psi$ if ψ is a p-form on M, $p > 0$, $n = \dim M$, and $\delta f = 0$ if f is a function. Show that δ is the formal adjoint of d with respect to the L^2 inner product.

Define the Laplace–Beltrami operator Δ and state the Hodge decomposition theorem. Show that if $\Delta\psi = \lambda\psi$ for some real number λ and some p-form $\psi \neq 0$ ($p \geqslant 0$), then $\lambda \geqslant 0$.

END OF PAPER