UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 7 June, 2011 9:00 am to 11:00 am

PAPER 12

COMBINATORICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State and prove the LYM inequality.

A set system $\mathcal{A} \subset \mathcal{P}([n])$ has the property that for any distinct $A, B \in \mathcal{A}$ we have $|A - B| > 1$ and $|B - A| > 1$.

By considering the shadow of $\mathcal{A}$, show that

$$\sum_{r=1}^{n} \frac{r|\mathcal{A}_r|}{\binom{n}{r-1}} \leqslant 1$$

where as usual $\mathcal{A}_r$ denotes $\{A \in \mathcal{A} : \ |A| = r\}$.

Using the fact that $(n+1)\binom{n}{r-1} = r\binom{n+1}{r}$, deduce from this that

$$|\mathcal{A}| \leqslant \frac{1}{n+1}\binom{n+1}{\lfloor (n+1)/2 \rfloor}.$$

By considering the sum of the elements of a set $A \in [n]^{(\lfloor n/2 \rfloor)}$, show that there exists a set system $\mathcal{A} \subset \mathcal{P}([n])$ satisfying the above condition with

$$|\mathcal{A}| \geqslant \frac{1}{n}\binom{n}{\lfloor n/2 \rfloor}.$$

2

State the vertex-isoperimetric inequality in the discrete cube (Harper's Theorem). Explain carefully how the Kruskal-Katona Theorem may be deduced from Harper's Theorem.

State the Erdős-Ko-Rado Theorem, and give two proofs: one using the Kruskal-Katona Theorem and one using cyclic orderings.

Let $\mathcal{A}, \mathcal{B} \subset [n]^{(r)}$, where $r \leqslant n/2$. Show that if $|\mathcal{A}|, |\mathcal{B}| > \binom{n-1}{r-1}$ then there exist $A \in \mathcal{A}$ and $B \in \mathcal{B}$ with $A \cap B = \emptyset$.

3

State and prove the vertex-isoperimetric inequality in the grid $[k]^n$.

[*You may assume that the theorem you are proving holds in the two-dimensional grid* $[k]^2$.]

Which of the following are always true and which can be false? Give proofs or counterexamples as appropriate.

(i) If k is even and A is a subset of $[k]^2$ of size $k^2/2$ then the boundary of A has size at least k.

(ii) If k is even and A is a subset of $[k]^3$ of size $k^3/2$ then the boundary of A has size at least $k^2 - 100k$.

4

State and prove the Uniform Covers Theorem.

State and prove the Bollobás-Thomason Box Theorem.

Let S be a body in $\mathbb{R}^4$ of volume 1 such that $|S_{123}| = |S_{124}| = |S_{134}| = 1$. What are the possible values of $|S_{234}|$?

END OF PAPER