UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 13 June, 2011 1:30 pm to 3:30 pm

PAPER 11

EXTREMAL GRAPH THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $k \geqslant 1$ be a natural number. Prove that, for n sufficiently large, $ex(n, C_{2k+1}) = \lfloor \frac{n^2}{4} \rfloor$.

[*You may assume the Erdős-Stone-Simonovits theorem provided it is stated correctly.*]

Is it true that for any graph H of chromatic number 3 there exists a constant c_H such that $ex(n, H) \leqslant \frac{n^2}{4} + c_H n$?

2

State and prove Szemerédi's regularity lemma.

Deduce the triangle removal lemma.

3

Let H be a bipartite graph between two sets U and V such that the degree of every vertex in V is at most Δ. Prove that there exists a constant c, depending on H, such that $ex(n, H) \leqslant cn^{2-\frac{1}{\Delta}}$.

Show that if there is only one vertex of degree Δ in V and all other vertices in V have degree at most $\frac{\Delta}{8}$, then there exists a constant c', depending on H, such that $ex(n, H) \leqslant c' n^{2-\frac{2}{\Delta}}$.

4

Given an r-uniform hypergraph $\mathcal{G}$ on n vertices, let N_s be the number of copies of $K_s^{(r)}$ in $\mathcal{G}$. Prove that

$$N_{s+1} \geqslant \frac{s^2 N_s}{(s-r+1)(s+1)} \left(\frac{N_s}{N_{s-1}} - \frac{(r-1)(n-s)+s}{s^2} \right),$$

provided $N_{s-1} \neq 0$. Deduce that

$$N_s \geqslant N_{s-1} \frac{r^2 \binom{s}{r}}{s^2 \binom{n}{r-1}} (e(\mathcal{G}) - F(n,s,r)),$$

where $F(n,s,r) = r^{-1}((n-r+1) - \binom{s-1}{r-1}^{-1}(n-s+1))\binom{n}{r-1}$. By considering an appropriate construction, conclude that

$$1 - \left(\frac{r-1}{s-1}\right)^{r-1} \leqslant \pi(K_s^{(r)}) \leqslant 1 - \binom{s-1}{r-1}^{-1}.$$

END OF PAPER